3.4. Actions, orbits and point stabilizers (handout)

In this handout we gather a list of examples of group actions and determine the orbits and the stabilizers.

3.4.1. G acting on G/H. Let G be a group, H a subgroup. The regular action of G on itself induces an action on the subsets of G (see Problem Set 6).

- Let $C = xH$ be a coset in G/H and let $g \in G$. Then gC is also a coset: $gC = g(xH) = (gx)H$. Accordingly the subset $G/H \subset P(G)$ is *invariant* and we can *restrict* the action of G to get an action on the invariant subset G/H.

(1) Orbits: for any two cosets xH, yH let $g = yx^{-1}$. Then $g(xH) = yx^{-1}xH = yH$ so there is only one orbit.
  - We say the action is *transitive*.

(2) Stabilizers: $\{g \mid gxH = xH\} = \left\{g \mid gxHx^{-1} = xHx^{-1}\right\} = \left\{g \mid g \in xHx^{-1}\right\} = xHx^{-1}$, so $\mathrm{Stab}_G(xH) = xHx^{-1}$ – in other words, the point stabilizers are exactly the *conjugates* of H.

PROPOSITION 181. *Let G act on X. For $x \in X$ let $H = \mathrm{Stab}_G(x)$ and let $f\colon G/H \to \mathcal{O}(x)$ be the bijection $f(gH) = gx$ of Proposition 176.* Then f is a map of G-sets*: for all $g \in G$ and coset $C \in G/H$ we have*

$$f(g \cdot C) = g \cdot f(C)$$

where on the left we have the action of g on $C \in G/H$ and on the left we have the action of g on $f(C) \in \mathcal{O}(x) \subset X$.

3.4.2. $\mathrm{GL}_n(\mathbb{R})$ acting on $\mathbb{R}^n$.

- For a matrix $g \in G = \mathrm{GL}_n(\mathbb{R})$ and vector $\underline{v} \in \mathbb{R}^n$ write $g \cdot \underline{v}$ for the matrix-vector product. This is an action (linear algebra).

(1) Orbits: We know that for all g, $g\underline{0} = \underline{0}$ so $\{\underline{0}\}$ is one orbit. For all other non-zero vectors we have:

CLAIM 182. Let V be a vector space, $\underline{u}, \underline{v} \in V$ be two non-zero vectors. Then there is a linear map $g \in \mathrm{GL}(V)$ such that $g\underline{u} = \underline{v}$.

We need a fact from linear algebra

FACT 183. *Let V, W be vector spaces and let $\{\underline{u}_i\}_{i \in I}$ be a basis of V. Let $\{\underline{w}_i\}_{i \in I}$ be any vectors in W. Then there is a unique linear map $f\colon V \to W$ such that $f(\underline{u}_i) = \underline{w}_i$.*

PROOF OF CLAIM. Complete $\underline{u}, \underline{v}$ to a bases $\{\underline{u}_i\}_{i \in I}, \{\underline{v}_i\}_{i \in I}$ ($\underline{u}_1 = \underline{u}$, $\underline{v}_1 = \underline{v}$). There is a unique linear map $g\colon V \to V$ such that $g\underline{u}_i = \underline{v}_i$ (because $\{\underline{u}_i\}$ is a basis) and similarly a unique map $h\colon V \to V$ such that $h\underline{v}_i = \underline{u}_i$. But then for all i we have $(gh)\underline{v}_i = \underline{v}_i = \mathrm{Id}\,\underline{v}_i$ and $(hg)\underline{u}_i = \underline{u}_i = \mathrm{Id}\,\underline{u}_i$, so by the uniqueness prong of the Fact we have $gh = \mathrm{Id} = hg$ and g is invertible, that is $g \in \mathrm{GL}(V)$. □

(2) Stabilizers: clearly all matrices stabilize zero. For other vectors we compute:

$$\mathrm{Stab}_{\mathrm{GL}_n(\mathbb{R})}(\underline{e}_n) = \left\{ g \mid g \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\} = \left\{ g = \begin{pmatrix} h & 0 \\ \underline{u} & 1 \end{pmatrix} \mid h \in \mathrm{GL}_{n-1}(\mathbb{R}), \underline{u} \in \mathbb{R}^{n-1} \right\}.$$

(more precisely the stabilizer consists of all matrices $g = \begin{pmatrix} h & 0 \\ \underline{u} & 1 \end{pmatrix}$ where $h \in M_{n-1}(\mathbb{R})$, but it is not hard to show that in this case g is invertible iff h is).

EXERCISE 184. Show that the block-diagonal matrices $M = \left\{ \begin{pmatrix} h & 0 \\ \underline{0} & 1 \end{pmatrix} \mid h \in \mathrm{GL}_{n-1}(\mathbb{R}) \right\}$ are a subgroup of $\mathrm{GL}_n(\mathbb{R})$ isomorphic to $\mathrm{GL}_{n-1}(\mathbb{R})$. Show that the matrices $N = \left\{ \begin{pmatrix} I_{n-1} & 0 \\ \underline{u} & 1 \end{pmatrix} \mid \underline{u} \in \mathbb{R}^{n-1} \right\}$ are a subgroup isomorphic to $(\mathbb{R}^{n-1}, +)$. Show that $\mathrm{Stab}_{\mathrm{GL}_n(\mathbb{R})}(\underline{e}_n)$ is the semidirect product $M \ltimes N$.

3.4.3. $\mathrm{GL}_n(\mathbb{R})$ acting on pairs of vectors (assume $n \geq 2$ here).

EXERCISE 185. If G acts on X and G acts on Y then setting $g \cdot (x, y) = (g \cdot x, g \cdot y)$ gives anthe action action of G on $X \times Y$.

We study the example where $G = \mathrm{GL}_n(\mathbb{R})$ and $X = Y = \mathbb{R}^n$.

(1) Orbits:
  (a) Clearly $(\underline{0}, \underline{0})$ is a fixed point of the action.
  (b) If $\underline{u}, \underline{v} \neq \underline{0}$ the previous discussion constructed g such that $g\underline{u} = \underline{v}$ and hence $g \cdot (\underline{u}, \underline{0}) = (\underline{v}, \underline{0})$ and $g \cdot (\underline{0}, \underline{u}) = (\underline{0}, \underline{v})$. Since $G \cdot (\underline{u}, \underline{0}) \subset \mathbb{R}^n \times \{\underline{0}\}$, we therefore get two more orbits: $\{(\underline{u}, \underline{0}) \mid \underline{u} \neq 0\}$ and $\{(\underline{0}, \underline{u}) \mid \underline{u} \neq 0\}$.
  (c) We now need to understand when there is g such that $g \cdot (\underline{u}_1, \underline{u}_2) = (\underline{v}_1, \underline{v}_2)$ when all vectors are nonzero. When studying the action on $\mathbb{R}^n$ itself we saw that if the pairs $\{\underline{u}_1, \underline{u}_2\}$, $\{\underline{v}_1, \underline{v}_2\}$ are each linearly independent then completing both to bases will provide such g. Conversely, if $\{\underline{u}_1, \underline{u}_2\}$ are independent then so are $\{g\underline{u}_1, g\underline{u}_2\}$ for any invertible g (g preserves the vector space structure hence linear algebra properties like linear independence). We therefore have an orbit

$$\{(\underline{u}_1, \underline{u}_2) \mid \text{ the vectors are linearly independent}\}.$$

  (d) The case of linear dependence remains, so we need to consider the orbit of $(\underline{u}_1, \underline{u}_2)$ where both are non-zero and $\underline{u}_2 = a\underline{u}_1$ for some scalar a, necessarily non-zero. In that case $g \cdot (\underline{u}_1, \underline{u}_2) = (g\underline{u}_1, g(a\underline{u}_1)) = (g\underline{u}_1, a(g\underline{u}_1))$ so the orbit of $(\underline{u}_1, a\underline{u}_1)$ is contained in

$$\{(\underline{v}, a\underline{v}) \mid \underline{v} \neq \underline{0}\}.$$

  Conversely, this is an orbit because if $\underline{u}_1, \underline{v}$ is are both non-zero there is g for which $g\underline{u}_1 = \underline{v}$ and then $g \cdot (\underline{u}_1, a\underline{u}_1) = (\underline{v}, a\underline{v})$.
  Summary: the orbits are $\{(\underline{0}, \underline{0})\}$, $\{(\underline{u}, \underline{0}) \mid \underline{u} \neq 0\}$,$\{(\underline{0}, \underline{u}) \mid \underline{u} \neq 0\}$, $\{(\underline{u}_1, \underline{u}_2) \mid \dim \mathrm{Span}_F \{\underline{u}_1, \underline{u}_2\} =$ and for each $a \in F^\times$ the set $\{(\underline{u}_1, a\underline{u}_1) \mid \underline{u}_1 \neq \underline{0}\}$.

(2) Point stabilizers:
  (a) $(\underline{0}, \underline{0})$ is fixed by the whole group.

(b) $g(\underline{u},\underline{0}) = (\underline{u},\underline{0})$ iff $g\underline{u} = \underline{u}$, so this is the case solved before. Similarly for $g \cdot (\underline{u}, a\underline{u}) = (\underline{u}, a\underline{u})$ which holds iff $g\underline{u} = \underline{u}$.

(c) $g\left(\underline{e}_{n-1}, \underline{e}_n\right) = \left(\underline{e}_{n-1}, \underline{e}_n\right)$ holds iff the last two columns of g are $\underline{e}_{n-1}, \underline{e}_n$ so

$$\mathrm{Stab}_{\mathrm{GL}_n(\mathbb{R})}\left(\underline{e}_{n-1}, \underline{e}_n\right) = \left\{ g = \begin{pmatrix} h & 0 \\ y & I_2 \end{pmatrix} \mid h \in \mathrm{GL}_{n-2}(\mathbb{R}), y \in M_{2,n-2}(\mathbb{R}) \right\}.$$

EXERCISE 186. Show that the block-diagonal matrices $M = \left\{ \begin{pmatrix} h & \underline{0} \\ \underline{0} & I_2 \end{pmatrix} \mid h \in \mathrm{GL}_{n-2}(\mathbb{R}) \right\}$ are a subgroup of $\mathrm{GL}_n(\mathbb{R})$ isomorphic to $\mathrm{GL}_{n-2}(\mathbb{R})$. Show that the matrices $N = \left\{ \begin{pmatrix} I_{n-2} & \underline{0} \\ y & 1 \end{pmatrix} \mid y \in M_{2,n-2}(\mathbb{R}) \right\} \simeq$ are a subgroup isomorphic to $\left(\mathbb{R}^{2(n-2)}, +\right)$. Show that $\mathrm{Stab}_{\mathrm{GL}_n(\mathbb{R})}\left(\underline{e}_{n-1}, \underline{e}_n\right)$ is the semidirect product $M \ltimes N$.

3.4.4. $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{PGL}_n(\mathbb{R})$ acting on $\mathbb{P}^{n-1}(\mathbb{R})$.

DEFINITION 187. Write $\mathbb{P}^{n-1}(\mathbb{R})$ for the set of 1-dimensional subspaces of $\mathbb{R}^n$ (this set is called "projective space of dimension $n-1$").

- Let $L \in \mathbb{P}^{n-1}(\mathbb{R})$, so that L is a line in $\mathbb{R}^n$ and let $g \in \mathrm{GL}_n(\mathbb{R})$. Then $g(L) = \{g\underline{v} \mid \underline{v} \in L\}$ is also a line (the image of a subspace is a subspace, and invertible linear maps preserve dimension), and this defines an action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{P}^{n-1}(\mathbb{R})$ (a restriction of the action of $\mathrm{GL}_n(\mathbb{R})$ on all subsets of $\mathbb{R}^n$ to the set of subsets which are lines).

(1) The action is transitive: suppose $L = \mathrm{Span}\{\underline{u}\}$ and $L' = \mathrm{Span}\{\underline{v}\}$ for some non-zero vectors $\underline{u}, \underline{v}$. Then any element g such that $g\underline{u} = \underline{v}$ will also map $gL = L'$.

(2) Suppose $L = \mathrm{Span}\{\underline{e}_n\}$. Then $gL = L$ means $g\underline{e}_n$ spans L, so $g\underline{e}_n = a\underline{e}_n$ for some non-zero a. It follows that

$$\mathrm{Stab}_{\mathrm{GL}_n(\mathbb{R})}(F \cdot \underline{e}_n) = \left\{ g = \begin{pmatrix} h & \underline{0} \\ \underline{u} & a \end{pmatrix} \mid h \in \mathrm{GL}_{n-1}(\mathbb{R}), a \in \mathbb{R}^\times \, \underline{u} \in \mathbb{R}^{n-1} \right\}.$$

- Repeat Exercize 184 from before, now with $M = \left\{ \begin{pmatrix} h & \underline{0} \\ \underline{0} & a \end{pmatrix} \mid h \in \mathrm{GL}_{n-1}(\mathbb{R}), a \in \mathbb{R}^\times \right\} \simeq \mathrm{GL}_{n-1}(\mathbb{R}) \times \mathbb{R}^\times$.

This can be generalized. For the same reason as for lines, the group $\mathrm{GL}_n(\mathbb{R})$ acts on the *Grassmannian*

$$\mathrm{Gr}(n,k) = \{L \subset \mathbb{R}^n \mid L \text{ is a subspace and } \dim_{\mathbb{R}} L = k\}.$$

The action is still transitive (for any L, L', take bases $\{\underline{u}_i\}_{i=1}^k \subset L$, $\{\underline{v}_i\}_{i=1}^k \subset L'$, complete both to bases of $\mathbb{R}^n$ and get a map), and the stabilizer will have the form $M \ltimes N$ with $M \simeq \mathrm{GL}_{n-k}(\mathbb{R}) \times \mathrm{GL}_k(\mathbb{R})$ and $N \simeq \left(M_{k,n-k}(\mathbb{R}), +\right)$.

3.4.5. $\mathrm{O}(n)$ acting on $\mathbb{R}^n$. Let the orthogonal group $\mathrm{O}(n) = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid g^t g = \mathrm{Id}\}$ act on $\mathbb{R}^n$.

- This a different kind of *restriction* – we restrict the action of $\mathrm{GL}_n(\mathbb{R})$ to a subgroup, but the set is still the whole of $\mathbb{R}^n$.

(1) Orbits: we know that if $g \in \mathrm{O}(n)$ and $\underline{v} \in \mathbb{R}^n$ then $\|g\underline{v}\| = \|\underline{v}\|$. Conversely, for each $a \geq 0$ $\{\underline{v} \in \mathbb{R}^n \mid \|vv\| = a\}$ is an orbit. When $a = 0$ this is clear (just the zero vector) and otherwise let $\underline{u}, \underline{v}$ both have norm a. Then $\underline{u}_1 = \frac{1}{a} vu$, $\underline{v}_1 = \frac{1}{a}\underline{v}$ are both unit vectors which

we can separately complete to orthonormal bases $\{\underline{u}_i\}, \{\underline{v}_i\}$ respectively. Then the unique invertible linear map $g \in \mathrm{GL}_n(\mathbb{R})$ such that $g\underline{u}_i = \underline{v}_i$ is orthogonal (linear algebra exercize). We thus obtain $g \in \mathrm{O}(n)$ such that $g\underline{u}_1 = \underline{v}_1$ and hence $g\underline{u} = g(a\underline{u}_1) = ag\underline{u}_1 = a\underline{v}_1 = \underline{v}$.

3.4.6. $\mathrm{Isom}(\mathbb{R}^n)$ **acting on** $\mathbb{R}^n$**.** Let $\mathrm{Isom}(\mathbb{R}^n)$ be the *Euclidean group*: the group of all *rigid motions* of $\mathbb{R}^n$ (maps $f\colon \mathbb{R}^n \to \mathbb{R}^n$ which preserve distance, in that $\|f(\underline{u}) - f(\underline{v})\| = \|\underline{u} - \underline{v}\|$).

(1) The action is transitive: for any fixed $\underline{a} \in \mathbb{R}^n$ the *translation* $T_{\underline{a}}\underline{x} = \underline{x} + \underline{a}$ preserves distances, making it an element of $\mathrm{Isom}(\mathbb{R}^n)$, and for all $\underline{u}, \underline{v}$ we have $T_{\underline{v}-\underline{u}}(\underline{u}) = \underline{v}$.

(2) The point stabilizer of zero is exactly the orthogonal group!

PROOF. We know that orthogonal maps preserve distances. Conversely let $f \in \mathrm{Isom}(\mathbb{R}^n)$ satisfy $f(\underline{0}) = \underline{0}$. Then f preserves distance from the origin:

$$\|f(\underline{x})\| = \|f(\underline{x}) - \underline{0}\| = \|f(\underline{x}) - f(\underline{0})\| = \|\underline{x} - \underline{0}\| = \|\underline{x}\|;$$

the difficulty is to show that f is a linear map. To showing that f preserves inner products first note that since $\|\underline{x} - \underline{y}\|^2 = \|\underline{x}\|^2 + \|\underline{y}\|^2 - 2\langle \underline{x}, \underline{y}\rangle$ we have the *polarization identity*

$$\langle \underline{x}, \underline{y}\rangle = \frac{1}{2}\left[\|\underline{x}\|^2 + \|\underline{y}\|^2 - \|\underline{x} - \underline{y}\|^2\right].$$

Thus

$$\begin{aligned}\langle f(\underline{x}), f(\underline{y})\rangle &= \frac{1}{2}\left[\|f(\underline{x})\|^2 + \|f(\underline{y})\|^2 - \|f(\underline{x}) - f(\underline{y})\|^2\right]\\ &= \frac{1}{2}\left[\|\underline{x}\|^2 + \|\underline{y}\|^2 - \|\underline{x} - \underline{y}\|^2\right]\\ &= \langle \underline{x}, \underline{y}\rangle .\end{aligned}$$

Next let $\{\underline{e}_i\}_{i=1}^n$ be the standard orthonormal basis. Since f preserves inner products, $\underline{u}_i = f(\underline{e}_i)$ also form an orthonormal basis, and there is a unique linear map $g \in \mathrm{O}(n)$ such that $g\underline{e}_i = \underline{u}_i$. We conclude by showing that $f = g$. For this let $\underline{x} \in \mathbb{R}^n$ and let $a_i = \langle \underline{x}, \underline{e}_i\rangle$. Then $\underline{x} = \sum_i a_i \underline{e}_i$ and since

$$\langle f(\underline{x}), \underline{u}_i\rangle = \langle f(\underline{x}), f(\underline{e}_i)\rangle = \langle \underline{x}, \underline{e}_i\rangle = a_i$$

we have

$$f(\underline{x}) = \sum_i a_i \underline{u}_i = \sum_i a_i g\underline{e}_i = g\left(\sum_i a_i \underline{e}_i\right) = g\underline{x}.$$

□

EXERCISE 188. Let $V = \left\{T_{\underline{a}} \mid \underline{a} \in \mathbb{R}^n\right\} \subset \mathrm{Isom}(\mathbb{R}^n)$ be the group of translations. This is a subgroup isomorphic to $\mathbb{R}^n$, and $\mathrm{Isom}(\mathbb{R}^n)$ is the semidirect product $\mathrm{O}(n) \ltimes V$.

EXERCISE 189. The orbits of $\mathrm{Isom}(\mathbb{R}^n)$ on the space of pairs $\mathbb{R}^n \times \mathbb{R}^n$ are exactly the sets $D_a = \left\{(\underline{x}, \underline{y}) \mid \|\underline{x} - \underline{y}\| = a\right\}$ $(a \geq 0)$.