

## Math 412: Problem Set 2 (due 21/9/2016)

### Practice

- P1 Let  $\{V_i\}_{i \in I}$  be a family of vector spaces, and let  $A_i \in \text{End}(V_i) = \text{Hom}(V_i, V_i)$ .
- Show that there is a unique element  $\bigoplus_{i \in I} A_i \in \text{End}(\bigoplus_{i \in I} V_i)$  whose restriction to the image of  $V_i$  in the sum is  $A_i$ .
  - Carefully show that the matrix of  $\bigoplus_{i \in I} A_i$  in an appropriate basis is block-diagonal.
- P2 Construct a vector space  $W$  and three subspaces  $U, V_1, V_2$  such that  $W = U \oplus V_1 = U \oplus V_2$  (internal direct sums) but  $V_1 \neq V_2$ .

### Direct sums

- Give an example of  $V_1, V_2, V_3 \subset W$  where  $V_i \cap V_j = \{0\}$  for every  $i \neq j$  yet the sum  $V_1 + V_2 + V_3$  is not direct.
- Let  $\{V_i\}_{i=1}^r$  be subspaces of  $W$  with  $\sum_{i=1}^r \dim(V_i) > (r-1) \dim W$ . Show that  $\bigcap_{i=1}^r V_i \neq \{0\}$ .
- (Diagonability)
  - Let  $T \in \text{End}(V)$ . For each  $\lambda \in F$  let  $V_\lambda = \text{Ker}(T - \lambda)$  be the corresponding eigenspace. Let  $\text{Spec}_F(T) = \{\lambda \in F \mid V_\lambda \neq \{0\}\}$  be the set of eigenvalues of  $T$ . Show that the sum  $\sum_{\lambda \in \text{Spec}_F(T)} V_\lambda$  is direct (the sum equals  $V$  iff  $T$  is diagonalizable).
  - Show that a square matrix  $A \in M_n(F)$  is diagonalizable over  $F$  iff there exist  $n$  one-dimensional subspaces  $V_i \subset F^n$  such  $F^n = \bigoplus_{i=1}^n V_i$  and  $A(V_i) \subset V_i$  for all  $i$ .

### Quotients

- Write  $M_n(F)$  for the space of  $n \times n$  matrices with entries in  $F$ . Let  $\mathfrak{sl}_n(F) = \{A \in M_n(F) \mid \text{Tr} A = 0\}$  and let  $\mathfrak{pgl}_n(F) = M_n(F)/F \cdot I_n$  (matrices modulo scalar matrices). Suppose that  $n$  is invertible in  $F$  (equivalently, that the characteristic of  $F$  does not divide  $n$ ). Show that the quotient map  $M_n(F) \rightarrow \mathfrak{pgl}_n(F)$  restricts to an isomorphism  $\mathfrak{sl}_n(F) \rightarrow \mathfrak{pgl}_n(F)$ .
- Recall our axiom that every vector space has a basis.
  - Show<sup>1</sup> that every linearly independent set in a vector space is contained in a basis.
  - Let  $U \subset W$ . Show that there exists another subspace  $V$  such that  $W = U \oplus V$ .
  - Let  $W = U \oplus V$ , and let  $\pi: W \rightarrow W/U$  be the quotient map. Show that the restriction of  $W$  to  $V$  is an isomorphism. Conclude that if  $U \oplus V_1 \simeq U \oplus V_2$  then  $V_1 \simeq V_2$  (c.f. problem P2)
- (Structure of quotients) Let  $V \subset W$  with quotient map  $\pi: W \rightarrow W/V$ .
  - Show that mapping  $U \mapsto \pi(U)$  gives a bijection between (1) the set of subspaces of  $W$  containing  $V$  and (2) the set of subspaces of  $W/V$ .
  - (The universal property) Let  $Z$  be another vector spaces. Show that  $f \mapsto f \circ \pi$  gives a linear bijection  $\text{Hom}(W/V, Z) \rightarrow \{g \in \text{Hom}(W, Z) \mid V \subset \text{Ker } g\}$ .

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<sup>1</sup>Without using any form of transfinite induction

**Extra credit**

7. For  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  the *Lipschitz constant* of  $f$  is the (possibly infinite) number

$$\|f\|_{\text{Lip}} \stackrel{\text{def}}{=} \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|} \mid x, y \in \mathbb{R}^n, x \neq y \right\}.$$

Let  $\text{Lip}(\mathbb{R}^n) = \left\{ f: \mathbb{R}^n \rightarrow \mathbb{R} \mid \|f\|_{\text{Lip}} < \infty \right\}$  be the space of *Lipschitz functions*.

PRA Show that  $f \in \text{Lip}(\mathbb{R}^n)$  iff there is  $C$  such that  $|f(x) - f(y)| \leq C|x - y|$  for all  $x, y \in \mathbb{R}^n$ .

- (a) Show that  $\text{Lip}(\mathbb{R}^n)$  is a vector space.
- (b) Let  $\mathbb{1}$  be the constant function 1. Show that  $\|f\|_{\text{Lip}}$  descends to a function on  $\text{Lip}(\mathbb{R}^n)/\mathbb{R}\mathbb{1}$ .
- (c) For  $\bar{f} \in \text{Lip}(\mathbb{R}^n)/\mathbb{R}\mathbb{1}$  show that  $\|\bar{f}\|_{\text{Lip}} = 0$  iff  $\bar{f} = 0$ .

**Supplement: Infinite direct sums and products**

CONSTRUCTION. Let  $\{V_i\}_{i \in I}$  be a (possibly infinite) family of vector spaces.

- (1) The direct product  $\prod_{i \in I} V_i$  is the vector space whose underlying space is  $\{f: I \rightarrow \bigcup_{i \in I} V_i \mid \forall i: f(i) \in V_i\}$  with the operations of pointwise addition and scalar multiplication.
- (2) The direct sum  $\bigoplus_{i \in I} V_i$  is the subspace  $\{f \in \prod_{i \in I} V_i \mid \#\{i \mid f(i) \neq \mathbf{0}_{V_i}\} < \infty\}$  of finitely supported functions.

A. (Tedium)

- (a) Show that the direct product is a vector space
- (b) Show that the direct sum is a subspace.
- (c) Let  $\pi_i: \prod_{i \in I} V_i \rightarrow V_i$  be the projection on the  $i$ th coordinate ( $\pi_i(f) = f(i)$ ). Show that  $\pi_i$  are surjective linear maps.
- (d) Let  $\sigma_i: V_i \rightarrow \prod_{i \in I} V_i$  be the map such that  $\sigma_i(v)(j) = \begin{cases} v & j = i \\ \mathbf{0} & j \neq i \end{cases}$ . Show that  $\sigma_i$  are injective linear maps.

B. (Meat) Let  $Z$  be another vector space.

- (a) Show that  $\bigoplus_{i \in I} V_i$  is the internal direct sum of the images  $\sigma_i(V_i)$ .
- (b) Suppose for each  $i \in I$  we are given  $f_i \in \text{Hom}(V_i, Z)$ . Show that there is a unique  $f \in \text{Hom}(\bigoplus_{i \in I} V_i, Z)$  such that  $f \circ \sigma_i = f_i$ .
- (c) You are instead given  $g_i \in \text{Hom}(Z, V_i)$ . Show that there is a unique  $g \in \text{Hom}(Z, \prod_{i \in I} V_i)$  such that  $\pi_i \circ g = g_i$  for all  $i$ .

C. (What a universal property can do) Let  $S$  be a vector space equipped with maps  $\sigma'_i: V_i \rightarrow S$ , and suppose the property of 5(b) holds (for every choice of  $f_i \in \text{Hom}(V_i, Z)$  there is a unique  $f \in \text{Hom}(S, Z)$  ...)

- (a) Show that each  $\sigma'_i$  is injective (hint: take  $Z = V_j$ ,  $f_j$  the identity map,  $f_i = 0$  if  $i \neq j$ ).
- (b) Show that the images of the  $\sigma'_i$  span  $S$ .
- (c) Show that  $S$  is the internal direct sum of the  $S_i$ .
- (d) (There is only one direct sum) Show that there is a unique isomorphism  $\varphi: S \rightarrow \bigoplus_{i \in I} V_i$  such that  $\varphi \circ \sigma'_i = \sigma_i$  (hint: construct  $\varphi$  by assumption, and a reverse map using the existence part of 5(b); to see that the composition is the identity use the uniqueness of the assumption and of 5(b), depending on the order of composition).

D. Now let  $P$  be a vector space equipped with maps  $\pi'_i: P \rightarrow V_i$  such that 5(c) holds.

- (a) Show that  $\pi'_i$  are surjective.
- (b) Show that there is a unique isomorphism  $\psi: P \rightarrow \prod_{i \in I} V_i$  such that  $\pi_i \circ \psi = \pi'_i$ .

### Supplement: universal properties

- E. A *free abelian group* is a pair  $(F, S)$  where  $F$  is an abelian group,  $S \subset F$ , and (“universal property”) for any abelian group  $A$  and any (set) map  $f: S \rightarrow A$  there is a unique group homomorphism  $\tilde{f}: F \rightarrow A$  such that  $\tilde{f}(s) = f(s)$  for any  $s \in S$ . The size  $\#S$  is called the *rank* of the free abelian group.
- Show that  $(\mathbb{Z}, \{1\})$  is a free abelian group.
  - Show that  $(\mathbb{Z}^d, \{e_k\}_{k=1}^d)$  is a free abelian group.
  - Let  $(F, S), (F', S')$  be free abelian groups and let  $f: S \rightarrow S'$  be a bijection. Show that  $f$  extends to a unique isomorphism  $\tilde{f}: F \rightarrow F'$ .
  - Let  $(F, S)$  be a free abelian group. Show that  $S$  generates  $F$ .
  - Show that every element of a free abelian group has infinite order.

### Supplement: Lipschitz functions

DEFINITION. Let  $(X, d_X), (Y, d_Y)$  be metric spaces, and let  $f: X \rightarrow Y$  be a function. We say  $f$  is a *Lipschitz function* (or is “Lipschitz continuous”) if for some  $C$  and for all  $x, x' \in X$  we have

$$d_Y(f(x), f(x')) \leq C d_X(x, x').$$

Write  $\text{Lip}(X, Y)$  for the space of Lipschitz continuous functions, and for  $f \in \text{Lip}(X, Y)$  write  $\|f\|_{\text{Lip}} = \sup \left\{ \frac{d_Y(f(x), f(x'))}{d_X(x, x')} \mid x \neq x' \in X \right\}$  for its *Lipschitz constant*.

- F. (Analysis)
- Show that Lipschitz functions are continuous.
  - Let  $f \in C^1(\mathbb{R}^n; \mathbb{R})$ . Show that  $\|f\|_{\text{Lip}} = \sup \{ |\nabla f(x)| : x \in \mathbb{R}^n \}$ .
  - Generalize 7(a),(b),(c) to the case of  $\text{Lip}(X, \mathbb{R})$  where  $X$  is any metric space.
  - Show that  $\text{Lip}(X, \mathbb{R})/\mathbb{R}\mathbb{1}$  is complete for all metric spaces  $X$ .