Math 101 – SOLUTIONS TO WORKSHEET 33
TAYLOR SERIES AND DERIVATIVES

1. Manipulating power series: summing series

(1) Find $\sum_{n=1}^{\infty} \frac{1}{n2^n}$.

Solution: We know that $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$, with radius of convergence 1. We then have:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n2^n} &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{-1}{2}\right)^n = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(-\frac{1}{2}\right)^n \\ &= = -\log\left(1-\frac{1}{2}\right) = -\log\frac{1}{2} = \log 2\,.\end{aligned}$$

(2) Avatars of geometric series.

(a) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

Solution: Let $h(x) = \sum_{n=1}^{\infty} nx^n$. We see that

$$\begin{aligned}h(x) &= x\sum_{n=1}^{\infty} nx^{n-1} = x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=1}^{\infty} x^n \\ &= x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n = x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} \\ &= \frac{x}{(1-x)^2}\,.\end{aligned}$$

Now the radius of convergence of $\sum_{n=0}^{\infty} x^n$ is 1, so $\frac{1}{2}$ is in the domain of convergence and we conclude

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = \frac{4}{2} = 2\,.$$

(b) Express $\sum_{n=1}^{\infty} n^2 x^n$ as a *rational function* (ratio of polynomials).

Solution: Let $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. We see that

$$\begin{aligned}f'(x) &= \sum_{n=0}^{\infty} nx^{n-1} \\ xf'(x) &= \sum_{n=0}^{\infty} nx^n \\ \left(xf'(x)\right)' &= \sum_{n=0}^{\infty} n^2 x^{n-1} \\ x\left(xf'(x)\right) &= \sum_{n=0}^{\infty} n^2 x^n\,,\end{aligned}$$

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so that

$$\sum_{n=1}^{\infty} n^2 x^n = \sum_{n=0}^{\infty} n^2 x^n = x\left(x\left(\frac{1}{1-x}\right)'\right)' = x\left(\frac{x}{(1-x)^2}\right)'$$

$$= x\left(\frac{1}{(1-x)^2} + \frac{2x}{(1-x)^3}\right) = \frac{x\left((1-x)+2x\right)}{(1-x)^3} = \boxed{\frac{x(1+x)}{(1-x)^3}}.$$

(3) Find a simple formula for $\sum_{n=0}^{\infty} \frac{e^{nx}}{n!}$.

Solution: We know that $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ so setting $u = e^x$ we get $\sum_{n=0}^{\infty} \frac{1}{n!} e^{nx} = \sum_{n=0}^{\infty} \frac{1}{n!} (e^x)^n = e^{e^x}$.

2. Taylor Series

The *Taylor series* of $f(x)$ *centered at* c is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n .$$

(4) Find the MacLaurin ($c = 0$) series of $f(x) = e^x$.

Solution: For each n we have $f^{(n)}(x) = e^x$ so $f^{(n)}(0) = e^0 = 1$. The series is therefore

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} .}$$

(5) (Final 2014) Find the Taylor series $g(x) = \log x$ centered at $a = 2$, as well as its radius of convergence.

Solution: $g'(x) = \frac{1}{x}$, $g''(x) = -\frac{1}{x^2}$, $g^{(3)}(x) = \frac{1\cdot 2}{x^3}$, $g^{(4)}(x) = -\frac{1\cdot 2\cdot 3}{x^4}$, and in general $g^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$. So for $n \geq 1$ we have $g^{(n)}(2) = (-1)^{n-1}\frac{(n-1)!}{2^k}$ and the Taylor series is

$$\log 2 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)!}{2^n n!}(x-2)^n = \log 2 + \sum_{n=1}^{\infty} (-1)^n \frac{(n-1)!}{2^n n(n-1)!}(x-2)^n$$

$$= \log 2 + \sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{2^n n} .$$

For the radius of convergence we compute $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{2^{n+1}(n+1)} \Big/ \frac{(-1)^n}{2^n n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{2^n}{2^{n+1}} = \frac{1}{2} \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{2}$ so we have $R = 2$.

Solution: We have

$$\log x = \log(2 + (x-2)) = \log\left(2\left(1 + \frac{x-2}{2}\right)\right) = \log 2 + \log\left(1 + \frac{x-2}{2}\right) .$$

We know that $\log(1+u) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} u^n$ and it follows that

$$\log x = \log 2 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{x-2}{2}\right)^n .$$

The logarithm series converges for $-1 < u \leq 1$ so our series will converge for $-1 < \frac{x-2}{2} \leq 1$ that is $-2 < x - 2 \leq 2$ so the radius of convergence is 2.

(6) (Final 2014) Let $\sum_{n=0}^{\infty} A_n x^n$ be the MacLaurin series for e^{3x}. Find A_5.

Solution: Knowing that $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ we have $e^{3x} = \sum_{n=0}^{\infty} \frac{3^n}{n!} x^n$ so $A_5 = \frac{3^5}{5!}$.

(7) (Final 2013) Let $f(x) = x^2 \sin(x^3)$. Find $f^{11}(0)$.

Solution: We know that $\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots$ so

$$x^2 \sin(x^3) = x^2 \left(x^3 - \frac{x^9}{3!} + \cdots\right) = x^5 - \frac{x^{11}}{3!} + \cdots .$$

It follows that $\frac{f^{(11)}(0)}{11!} = \frac{1}{3!}$ so $f^{(11)}(0) = \frac{11!}{3!}$.