Math 101 – SOLUTIONS TO WORKSHEET 19
MORE WORK

(1) (Preliminary) A worker carries a 20kg bucket to the top of a 10m tall building. Half way up the worker picks up a second 20kg bucket. Calculate the total work done by the worker by adding the contributions from carrying each bucket separately.

Solution: The work done on the first bucket is $20\text{kg}\cdot 10\text{m}\cdot g = 200\cdot 9.8\text{J} = 1960\text{J}$. The work done on the second bucket is half as much, 980J. The total work is 2940J.

(2) (Quiz, 2015) A10m-long cable of mass 7kg is used to lift a bucket off the ground. How much work is needed to raise the entire cable to the height of 10m? Ignore the weight of the bucket, and use $g = 9.8\text{m}/\sec^2$ for the acceleration due to gravity.

Solution: ("Chop up the mass") We first figure out the work required to lift a piece of the cable of length $\mathrm{d}x$ which is initially at height x off the ground.
The mass per unit length of the cable is $\rho = \frac{7\text{kg}}{10\text{m}} = 0.7\frac{\text{kg}}{\text{m}}$ so the infinitesimal piece of the cable weights $0.7\,\mathrm{d}x$ kilograms. It will be lifted a distance $10 - x$ against gravity, so the work in lifting the piece of the chain is $(0.7\,\mathrm{d}x)\cdot g\cdot(10-x) = 0.7\cdot 9.8\cdot(10-x)\,\mathrm{d}x$. The work for lifting the whole chain is therefore:

$$\int_0^{10} 0.7\cdot 9.8\cdot(10-x)\,\mathrm{d}x = 0.7\cdot 9.8\cdot\left[10x - \frac{1}{2}x^2\right]_{x=0}^{x=10} = 0.7\cdot 9.8\cdot 50\text{J} = 343\text{J}\,.$$

Solution: (Clever choice of coordinates) Let $\rho = \frac{7\text{kg}}{10\text{m}} = 0.7\frac{\text{kg}}{\text{m}}$ be the mass density of the cable, $H = 10\text{m}$ be its length. Consider a part of the cable of length $\mathrm{d}z$, hanging height z below the roof. This part has mass $\rho\,\mathrm{d}z$ and will be lifted height z against standard gravity. The work in lifting the cable is therefore

$$\int_0^H \rho g z\,\mathrm{d}z = \rho g\left[\frac{1}{2}z^2\right]_0^H = \frac{1}{2}\rho g H^2 = \frac{1}{2}\cdot 0.7\cdot 9.8\cdot 100\text{J} = 343\text{J}\,.$$

Solution: ("Chop up the process') Consider the work done between cable having been raised z units of height and having been raised $z + \mathrm{d}z$ units of height. After lifting z units of height, the part of the cable still hanging has length $H - z$ and mass $\rho(H - z)$. It will be lifted a height $\mathrm{d}z$ against gravity, so the total work is

$$\int_0^H \rho(H-z)g\,\mathrm{d}z = \left[\rho H z - \frac{1}{2}\rho H^2\right]_{z=0}^{z=H} g = \frac{1}{2}\rho g H^2 = 343\text{J}\,.$$

(3) (Final, 2012) A tank in the shape of a hemispherical bowl of radius $R = 3\text{m}$ is full of water. It is to be emptied through an outlet extending $H = 2\text{m}$ above its top. Using the values $\rho = 1000\text{kg}/\text{m}^3$ for the density of water and $g = 9.8\text{m}/\text{s}^2$ for the acceleration due to gravity, find the work (in Joules) required to empty the tank completely. There is no need to simplify your answer but you must evaluate all integrals.

Solution: Slice the water into horizontal slabs perpendicular to the z-axis. Each slab has circular cross-section. The slab at height z above the bottom of the hemisphere is at height $R - z$ below the centre of the sphere, so the radius of the slab satisfies

$$r^2 + (R-z)^2 = R^2\,,$$

that is

$$r^2 = 2Rz - z^2\,.$$

The volume of the slab is therefore about $\pi r^2\Delta z = \pi(2Rz - z^2)\Delta z$ and the mass of the water in it is

$$\Delta m \approx \pi\rho\left(2Rz - z^2\right)\Delta z\,.$$

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This mass of water will be moved from the height z to the height $R+H$ against gravity, the required work is therefore

$$\begin{aligned}\Delta W &= \Delta m \cdot g \cdot (R+H-z) \\ &= \pi\rho g \left(2Rz - z^2\right)(R+H-z)\,\Delta z\,.\end{aligned}$$

The total work is therefore

$$\begin{aligned}\pi\rho g \int_{z=0}^{z=R} \left(2Rz - z^2\right)(R+H-z)\,\mathrm{d}z &= \pi\rho g \int_{z=0}^{z=R} \left(2Rz(R+H) - z^2(R+H) - 2Rz^2 + z^3\right)\mathrm{d}z \\ &= \pi\rho g \int_{z=0}^{z=R} \left(2R(R+H)z - (3R+H)z^2 + z^3\right)\mathrm{d}z \\ &= \pi\rho g \left[R(R+H)z^2 - \frac{3R+H}{3}z^3 + \frac{z^4}{4}\right]_{z=0}^{z=R} \\ &= \pi\rho g \left((R+H)R^3 - (R+\frac{H}{3})R^3 + \frac{R^4}{4}\right) \\ &= \pi\rho g R^3 \left(\frac{R}{4} + \frac{2}{3}H\right) \\ &= 9,800\pi \cdot 27 \left(\frac{3}{4} + \frac{4}{3}\right) \mathrm{J} \\ &= \boxed{551,250\pi \mathrm{J}\,.}\end{aligned}$$

Solution: (Clever coordinates) Instead, let z measure depth below the surface of the water. Again we chop the water into horizontal slabs perpendicular to the z-axis. Then the slice at depth z has volume about $\pi r^2 \Delta z = \pi \left(R^2 - z^2\right)\Delta z$. This water will be moved distance $H+z$ against gravity, so the total work is

$$\begin{aligned}\int_{z=0}^{R} \pi\rho g \left(R^2 - z^2\right)(H+z)\,\mathrm{d}z &= \pi\rho g \int_{z=0}^{R} \left(R^2 H + R^2 z - z^2 H - z^3\right)\mathrm{d}z \\ &= \pi\rho g \left[R^2 H z + \frac{R^2}{2}z^2 - \frac{H}{3}z^3 - \frac{z^4}{4}\right]_{z=0}^{z=R} \\ &= \pi\rho g \left[R^3 H + \frac{R^4}{2} - \frac{H}{3}R^3 - \frac{R^4}{4}\right] \\ &= \pi\rho g R^3 \left(\frac{2}{3}H + \frac{R}{4}\right) \\ &= 9,800\pi \cdot 27 \left(\frac{4}{3} + \frac{3}{4}\right) \mathrm{J}\,.\end{aligned}$$

(4) (Final, 2010) A colony of ants builds an anthill that is in the shape of a cone whose base, at ground level, is a circle of diameter 1ft and whose height is also 1ft. How much total work, in ftlb$s$, is done by the ants in building the anthill? For the density of sand, use the value 150lb/ft^3.

Solution: Consider a horizontal slice the cone at height x above the ground and of thickness dx. This slice has circular cross-section. Say the diameter is d. Let H be the height of the cone, D its diameter. By similar triangles, $\frac{d}{H-x} = \frac{D}{H}$ so that

$$d = D\frac{H-x}{H} = D\left(1 - \frac{x}{H}\right)\,.$$

Let ρ be the specific weight of the sand. Then this slab has weight $\rho\left(\frac{1}{4}\pi d^2\,\mathrm{d}x\right) = \frac{1}{4}\pi\rho D^2 \left(1 - \frac{h}{X}\right)^2 \mathrm{d}x$. The sand for this slab was lifted height x, so the work done on the slab is

$$\mathrm{d}W = \frac{1}{4}\pi\rho D^2 \left(1 - \frac{x}{H}\right)^2 x\,\mathrm{d}x$$

and the total work was

$$
\begin{aligned}
W &= \frac{1}{4}\pi\rho D^2 \int_0^H \left(1 - \frac{x}{H}\right)^2 x \,\mathrm{d}x \\
&= \frac{1}{4}\pi\rho D^2 \int_0^H \left(x - \frac{2x^2}{H} + \frac{x^3}{H^2}\right) \mathrm{d}x \\
&= \frac{1}{4}\pi\rho D^2 \left[\frac{1}{2}x^2 - \frac{2x^3}{3H} + \frac{x^4}{4H^2}\right]_{x=0}^{x=H} \\
&= \frac{1}{4}\pi\rho D^2 \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] H^2 \\
&= \frac{1}{4}\pi\rho D^2 \frac{1}{12} H^2 = \frac{\pi}{48}\rho D^2 H^2 \\
&= \frac{\pi}{48} 150 \mathrm{ftlb} = \frac{25\pi}{8} \mathrm{ftlb}\,.
\end{aligned}
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