Math 101 – SOLUTIONS TO WORKSHEET 18
WORK

In this course we use the approximate value $g = 9.8\text{m/s}^2$.

(1) (Preliminary) A worker carries a 20kg bucket to the top of a 10m tall building. Half way up the worker picks up a second 20kg bucket. Calculate the total work done by the worker: by adding the contributions from carrying each bucket separately.

Solution: The work done on the first bucket is $20\text{kg} \cdot 10\text{m} \cdot g = 200 \cdot 9.8\text{J} = 1960\text{J}$. The work done on the second bucket is half as much, 980J. The total work is 2940J.

(2) (Quiz, 2015, variant) A bucket weighting 20kg is pulled up the side of a 10m-tall building by a chain weighing 7kg. How much work is rqeuired to pull the bucket and chain in pulling the bucket and chain to the roof?

Solution: ("Chop up the mass") As to the bucket, the work required is 1960J as calculated above. For the chain, we figure out the work required to lift a piece of length $\mathrm{d}x$ at height x off the ground.

The mass per unit length of the chain is $\rho = \frac{7\text{kg}}{10\text{m}} = 0.7\frac{\text{kg}}{\text{m}}$ so the infinitesimal piece of the chain weights $0.7\,\mathrm{d}x$ kilograms. It will be lifted a distance $10 - x$ against gravity, so the work in lifting the piece of the chain is $(0.7\,\mathrm{d}x) \cdot g \cdot (10 - x) = 0.7 \cdot 9.8 \cdot (10 - x)\,\mathrm{d}x$. The work for lifting the whole chain is therefore:

$$\int_0^{10} 0.7 \cdot 9.8 \cdot (10 - x)\,\mathrm{d}x = 0.7 \cdot 9.8 \cdot \left[10x - \frac{1}{2}x^2\right]_{x=0}^{x=10} = 0.7 \cdot 9.8 \cdot 50\text{J} = 343\text{J}\,.$$

The work in lifting the bucket and the chain is therefore $1960\text{J} + 343\text{J} = 2303\text{J}$.

Solution: (Clever choice of coordinates) Let $\rho = \frac{7\text{kg}}{10\text{m}} = 0.7\frac{\text{kg}}{\text{m}}$ be the mass density of the chain, $H = 10\text{m}$ be the height of the building. Consider a part of the chain of length $\mathrm{d}z$, hanging height z below the roof. This part has mass $\rho\,\mathrm{d}z$ and will be lifted height z against standard gravity. The work in lifting the chain is therefore

$$\int_0^H \rho g z\,\mathrm{d}z = \rho g \left[\frac{1}{2}z^2\right]_0^H = \frac{1}{2}\rho g H^2 = \frac{1}{2} \cdot 0.7 \cdot 9.8 \cdot 100\text{J} = 343\text{J}\,.$$

Solution: ("Chop up the process') Consider the work done between lifting the chain z units of height and $z + \mathrm{d}z$ units of height. After we have lifted the chain z units of height, the part still hanging has length $H - z$ and mass $(\rho(H - z) + m)$ where m is the mass of the bucket. It will be lifted a height $\mathrm{d}z$ against gravity, so the total work is

$$\int_0^H (\rho(H - z) + m)\,g\,\mathrm{d}z = \left[\rho H z - \frac{1}{2}\rho H^2 + mz\right]_{z=0}^{z=H} g = \frac{1}{2}\rho g H^2 + mgH = 343\text{J} + 1960\text{J} = 2303\text{J}\,.$$

(3) (Final, 2012) A tank in the shape of a hemispherical bowl of radius $R = 3\text{m}$ is full of water. It is to be emptied through an outlet extending $H = 2\text{m}$ above its top. Using the values $\rho = 1000\text{kg/m}^3$ for the density of water and $g = 9.8\text{m/s}^2$ for the acceleration due to gravity, find the work (in Joules) required to empty the tank completely. There is no need to simplify your answer but you must evaluate all integrals.

Solution: Slice the water into horizontal slabs perpendicular to the z-axis. Each slab has circular cross-section. The slab at height z above the bottom of the hemisphere is at distance $R - z$ below the centre of the sphere, so the radius of the slab satisfies

$$r^2 + (R - z)^2 = R^2$$

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so

$$r^2 = 2Rz - z^2\,.$$

The volume of the slab is therefore about $\pi r^2 \Delta z = \pi(2Rz - z^2)\Delta z$; the mass of the water in it is therefore

$$\Delta m \approx \pi\rho\left(2Rz - z^2\right)\Delta z\,.$$

This mass of water will be moved from the height z to the height $R+H$ against gravity, the required work is therefore

$$\begin{aligned}\Delta W &= \Delta m \cdot g \cdot (R + H - z) \\ &= \pi\rho g\left(2Rz - z^2\right)(R + H - z)\,\Delta z\,.\end{aligned}$$

The total work is therefore

$$\begin{aligned}\pi\rho g\int_{z=0}^{z=R}\left(2Rz - z^2\right)(R+H-z)\,\mathrm{d}z &= \pi\rho g\int_{z=0}^{z=R}\left(2Rz(R+H) - z^2(R+H) - 2Rz^2 + z^3\right)\mathrm{d}z \\ &= \pi\rho g\int_{z=0}^{z=R}\left(2R(R+H)z - (3R+H)z^2 + z^3\right)\mathrm{d}z \\ &= \pi\rho g\left[R(R+H)z^2 - \frac{3R+H}{3}z^3 + \frac{z^4}{4}\right]_{z=0}^{z=R} \\ &= \pi\rho g\left((R+H)R^3 - (R+\frac{H}{3})R^3 + \frac{R^4}{4}\right) \\ &= \pi\rho g R^3\left(\frac{R}{4} + \frac{2}{3}H\right) \\ &= 9,800\pi\cdot 27\left(\frac{3}{4} + \frac{4}{3}\right)\mathrm{J}\,.\end{aligned}$$

Solution: Instead, let z measure depth below the surface of the water. Again we chop the water into horizontal slabs perpendicular to the z-axis. Then the slice at depth z has volume about $\pi r^2\Delta z = \pi\left(R^2 - z^2\right)\Delta z$. This water will be moved distance $H+z$ against gravity, so the total work is

$$\begin{aligned}\int_{z=0}^{R}\pi\rho g\left(R^2 - z^2\right)(H+z)\,\mathrm{d}z &= \pi\rho g\int_{z=0}^{R}\left(R^2H + R^2z - z^2H - z^3\right)\mathrm{d}z \\ &= \pi\rho g\left[R^2Hz + \frac{R^2}{2}z^2 - \frac{H}{3}z^3 - \frac{z^4}{4}\right]_{z=0}^{z=R} \\ &= \pi\rho g\left[R^3H + \frac{R^4}{2} - \frac{H}{3}R^3 - \frac{R^4}{4}\right] \\ &= \pi\rho g R^3\left(\frac{2}{3}H + \frac{R}{4}\right) \\ &= 9,800\pi\cdot 27\left(\frac{4}{3} + \frac{3}{4}\right)\mathrm{J}\,.\end{aligned}$$

(4) (Final, 2010) A colony of ants builds an anthill that is in the shape of a cone whose base, at ground level, is a circle of diameter 1ft and whose height is also 1ft. How much total work, in ftlb$s$, is done by the ants in building the anthill? For the density of sand, use the value 150lb/ft^3.

Solution: Consider a horizontal slice the cone at height x above the ground and of thickness dx. This slice has circular cross-section. Say the diameter is d. Let H be the height of the cone, D its diameter. By similar triangles, $\frac{d}{H-x} = \frac{D}{H}$ so that

$$d = D\frac{H-x}{H} = D\left(1 - \frac{x}{H}\right)\,.$$

Let ρ be the specific weight of the sand. Then this slab has weight $\rho\left(\frac{1}{4}\pi d^2\,\mathrm{d}x\right) = \frac{1}{4}\pi\rho D^2\left(1-\frac{h}{X}\right)^2\mathrm{d}x$. The sand for this slab was lifted height x, so the work done on the slab is

$$\mathrm{d}W = \frac{1}{4}\pi\rho D^2\left(1-\frac{x}{H}\right)^2 x\,\mathrm{d}x$$

and the total work was

$$\begin{aligned}
W &= \frac{1}{4}\pi\rho D^2\int_0^H\left(1-\frac{x}{H}\right)^2 x\,\mathrm{d}x \\
&= \frac{1}{4}\pi\rho D^2\int_0^H\left(x-\frac{2x^2}{H}+\frac{x^3}{H^2}\right)\mathrm{d}x \\
&= \frac{1}{4}\pi\rho D^2\left[\frac{1}{2}x^2-\frac{2x^3}{3H}+\frac{x^4}{4H^2}\right]_{x=0}^{x=H} \\
&= \frac{1}{4}\pi\rho D^2\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right]H^2 \\
&= \frac{1}{4}\pi\rho D^2\frac{1}{12}H^2 = \frac{\pi}{48}\rho D^2H^2 \\
&= \frac{\pi}{48}150\mathrm{ftlb}\,.
\end{aligned}$$