Math 101 – SOLUTIONS TO WORKSHEET 16
APPROXIMATE INTEGRATION

(1) (Final 2012) Let $I = \int_1^2 \frac{1}{x}\,\mathrm{d}x$.

(a) Write down Simpson's rule approximation for I using 4 points (call it S_4)

Solution: $S_4 = \frac{1}{12}\left(\frac{1}{1} + 4\frac{1}{5/4} + 2\frac{1}{3/2} + 4\frac{1}{7/4} + \frac{1}{2}\right)$.

It was not required to do the arithmetic, but for the record we note (since $210 = 2\cdot 3\cdot 5\cdot 7$):

$$\begin{aligned} S_4 &= \frac{1}{12}\left(1 + \frac{16}{5} + \frac{4}{3} + \frac{16}{7} + \frac{1}{2}\right) \\ &= \frac{1}{12}\cdot\frac{210 + 42\cdot 16 + 70\cdot 3 + 30\cdot 16 + 105}{210} \\ &= \frac{1677}{2520}. \end{aligned}$$

(b) Without computing I, find an upper bound for $|I - S_4|$. You may use the fact that if $\left|f^{(4)}(x)\right| \le K$ on $[a,b]$ then the error in the approximation with n points has magnitude at most $K(b-a)^5/180n^4$.

Solution: We have $f'(x) = -\frac{1}{x^2}$, $f^{(2)}(x) = \frac{2}{x^3}$, $f^{(3)}(x) = -\frac{6}{x^4}$ and $f^{(4)}(x) = \frac{24}{x^5}$. On the interval $[1,2]$, the function $\frac{24}{x^5}$ is decreasing so $\left|f^{(4)}(x)\right| \le \frac{24}{1} = 24$. It follows that the error is at most

$$\frac{24(2-1)^5}{180\cdot 4^4} = \frac{24}{180\cdot 256} = \frac{1}{60\cdot 32} = \frac{1}{1960}.$$

(2) (Final 2015) Consider $I = \int_0^2 (x-3)^5\,\mathrm{d}x$.

(a) Write down the Simpson's rule approximation to I with $n = 6$. You may leave your answers in calculator-ready form.

Solution: In each case $\Delta x = \frac{2}{6} = \frac{1}{3}$. The solution is therefore

$$I \approx \frac{1}{3\cdot 3}\left[(0-3)^5 + 4\left(\frac{1}{3} - 3\right)^5 + 2\left(\frac{2}{3} - 3\right)^5 + 4(1-3)^5 + 2\left(\frac{4}{3} - 3\right)^5 + 4\left(\frac{5}{3} - 3\right)^5 + (2-3)^5\right].$$

(b) Which method of approximating I results in a smaller error bound: the Midpoint Rule with $n = 100$ intervals, or Simpson's Rule with $n = 10$ intervals? Justify your answer. You may use the formulas $|E_{\mathrm{M}}| \le \frac{K(b-a)^3}{24n^2}$ and $|E_S| \le \frac{L(b-a)^5}{180n^4}$ where K is an upper bound for $|f''(x)|$ and L is an upper bound for $\left|f^{(4)}(x)\right|$.

Solution: Here $f''(x) = 5\cdot 4(x-3)^3$ and $f^{(4)}(x) = 5\cdot 4\cdot 3\cdot 2(x-3)$. For $0 \le x \le 2$, $|x-3| \le 3$ so we may take $K = 20\cdot 3^3 = 540$ and $L = 120\cdot 3 = 360$. Plugging in the formulas we get the estimates

$$\begin{aligned} |E_M| &\le \frac{540\cdot 2^3}{24\cdot 10^4} = \frac{180}{10^4} \\ |E_S| &\le \frac{360\cdot 2^5}{180\cdot 10^4} = \frac{64}{10^4}. \end{aligned}$$

We conclude that Simpson's rule gives the better error esimate.

(3) (Final 2008) Let $I = \int_0^1 \cos(x^2)\,\mathrm{d}x$. It can be shown that the fourth derivative of $\cos(x^2)$ has absolute value at most 60 on $[0,1]$. Find n such the Simpson's rule approximation to I using n points has error less than or equal to 0.001. You may use that that if $\left|f^{(4)}(t)\right| \le K$ for $a \le t \le b$ then error in using Simpson's rule to approximate $\int_a^b f(x)\,\mathrm{d}x$ has absolute value less than or equal to $K(b-a)^5/180n^4$.

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Solution: For $f(x) = \cos(x^2)$ we are given that $\left|f^{(4)}(x)\right| \leq 60$ for $1 \leq x \leq 2$, so we need n such that $\frac{60 \cdot (1-0)^5}{180n^4} \leq \frac{1}{1000}$, that is

$$\frac{1}{3n^4} \leq \frac{1}{1000}$$

which is the same as

$$n^4 \geq \frac{1000}{3}.$$

Now for $n = 6$ we have $6^4 = 36 \cdot 36 \geq 30 \cdot 30 = 900 > \frac{1000}{3}$ so $n = 6$ suffices.

(4) Let $I = \int_4^6 \sin(\sqrt{x})\,dx$. Find n such that estimating I using the midpoint rule and n points will have an error of at most $\frac{1}{3000}$. You may use that the absolute error in estimating $\int_a^b f(x)\,dx$ using the midpoint rule and n points is at most $K(b-a)^3/24n^2$ whenever $\left|f^{(2)}(x)\right| \leq K$ for $a \leq x \leq b$.

Solution: Let $f(x) = \sin(\sqrt{x})$. Then $f'(x) = \frac{1}{2\sqrt{x}}\cos(\sqrt{x})$ so $f''(x) = -\frac{1}{4x^{3/2}}\cos(\sqrt{x}) - \frac{1}{4x}\sin(\sqrt{x})$. For $4 \leq x \leq 6$ we have $\frac{1}{4x^{3/2}} \leq \frac{1}{4 \cdot 4^{3/2}} = \frac{1}{32}$ ($\frac{1}{x^{3/2}}$ is decreasing on this interval) and $\frac{1}{4x} \leq \frac{1}{4 \cdot 4} = \frac{1}{16}$ (for the same reason). Since $|\cos(\sqrt{x})|, |\sin(\sqrt{x})| \leq 1$ for all x, we have

$$\left|f^{(2)}(x)\right| \leq \frac{1}{32} + \frac{1}{16} = \frac{3}{32} \leq \frac{3}{30} = \frac{1}{10}$$

for all $4 \leq x \leq 6$. It follows that the error in the approximation is at most

$$\frac{1}{10} \cdot \frac{(6-4)^3}{24 \cdot n^2} = \frac{8}{240n^2} = \frac{1}{30n^2}.$$

For $n = 10$ the error would be at most $\frac{1}{30 \cdot 100} = \frac{1}{3000}$ so that is enough.