Math 101 – SOLUTIONS TO WORKSHEET 12
TRIGONOMETRIC SUBSTITUTION

1. Trig substitution

(1) (Final, 2014) Evaluate $\int \sqrt{4-x^2}\,\mathrm{d}x$

Solution: Let $x = 2\sin\theta$ so that $\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$. Then

$$\begin{aligned}\int \sqrt{4-x^2}\,\mathrm{d}x &= \int \sqrt{4-4\sin^2\theta}\,(2\cos\theta\,\mathrm{d}\theta)\\ &= 2\int \sqrt{4}\sqrt{1-\sin^2\theta}\cos\theta\,\mathrm{d}\theta = 4\int \cos^2\theta\,\mathrm{d}\theta\\ &= 4\int \frac{1+\cos(2\theta)}{2}\,\mathrm{d}\theta = 2\int \mathrm{d}\theta + \int \cos(2\theta)\,\mathrm{d}(2\theta)\\ &= 2\theta + \sin(2\theta) = 2\theta + 2\sin\theta\cos\theta + C\\ &= \boxed{2\arcsin\left(\frac{x}{2}\right) + x\sqrt{1-\frac{x^2}{4}} + C}.\end{aligned}$$

In the last row we used $\sin\theta = \frac{x}{2}$ to get $\theta = \arcsin\left(\frac{x}{2}\right)$, $\cos\theta = \sqrt{1-\sin^2\theta}$.

(2) (Final, 2013) Evaluate $\int_{-1}^{1} \frac{\mathrm{d}x}{(x^2+1)^3}$

Solution: Recalling $1+\tan^2\theta = \sec^2\theta$ let $x = \tan\theta$ for which $\mathrm{d}x = \sec^2\theta$. We also know that $\tan\left(\pm\frac{\pi}{4}\right) = \pm 1$ so the integral becomes:

$$\int_{x=-1}^{x=1} \frac{\mathrm{d}x}{(x^2+1)^3} = \int_{\theta=-\pi/4}^{\theta=\pi/4} \frac{\sec^2\theta}{,(\sec^2\theta)^3}\,\mathrm{d}\theta = \int_{\theta=-\pi/4}^{\theta=\pi/4} \cos^4\theta\,\mathrm{d}\theta\,.$$

We now use trig integral techniques, here the half-angle formula $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ to get

$$\begin{aligned}\int_{\theta=-\pi/4}^{\theta=\pi/4} \cos^4\theta\,\mathrm{d}\theta &= \int_{\theta=-\pi/4}^{\theta=\pi/4} \left(\frac{1+\cos(2\theta)}{2}\right)^2 \mathrm{d}\theta\\ &= \int_{\theta=-\pi/4}^{\theta=\pi/4} \left(\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1}{4}\cos^2(2\theta)\right)\mathrm{d}\theta\end{aligned}$$

The half-angle formula again gives $\cos^2(2\theta) = \frac{1+\cos(4\theta)}{2}$ so

$$\begin{aligned}&= \int_{\theta=-\pi/4}^{\theta=\pi/4} \left(\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{1+\cos(4\theta)}{8}\right)\mathrm{d}\theta\\ &= \left[\frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + \frac{1}{32}\sin(4\theta)\right]_{\theta=-\pi/4}^{\theta=\pi/4}\\ &= \frac{3}{16}\pi + \frac{1}{2}\sin(\frac{\pi}{2}) + \frac{1}{32}\sin(\pi) = \boxed{\frac{3}{16}\pi + \frac{1}{2}}.\end{aligned}$$

(3) (105 Final, 2012) Evaluate the indefinite integral $\int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x$

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Solution: Recalling $\tan^2\theta = \sec^2\theta - 1$ let $x = \frac{2}{5}\sec\theta$ for which $\mathrm{d}x = \frac{2}{5}\sec\theta\tan\theta$. Then $25x^2 - 4 = 4\sec^2\theta - 4 = 4\tan^2\theta$ and the integral becomes

$$\int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x = \int \frac{2\tan\theta}{\frac{2}{5}\sec\theta}\cdot\frac{2}{5}\sec\theta\tan\theta\,\mathrm{d}\theta = 2\int \tan^2\theta\,\mathrm{d}\theta\,.$$

We now use trig integral techniques, here we recall that $(\tan\theta)' = \sec^2\theta = \tan^2\theta + 1$ so that

$$\begin{aligned} 2\int \tan^2\theta\,\mathrm{d}\theta &= 2\int \left(\sec^2\theta + 1\right)\mathrm{d}\theta \\ &= 2\tan\theta + 2\theta + C. \end{aligned}$$

Finally, $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\frac{25}{4}x^2 - 1}$ and $\theta = \arccos\frac{2}{5x}$ so that

$$\begin{aligned} \int \frac{\sqrt{25x^2-4}}{x}\,\mathrm{d}x &= 2\sqrt{\frac{25x^2}{4} - 1} + 2\arccos\frac{2}{5x} + C \\ &= \sqrt{25x^2-4} + 2\arccos\frac{2}{5x} + C\,. \end{aligned}$$

2. COMPLETING THE SQUARE ETC

(4) (105 Final, 2014 + 101 Final, 2009) Convert $\int \left(3 - 2x - x^2\right)^{-3/2}\mathrm{d}x$ to a trigonometric integral.

Solution: We complete the square: $3 - 2x - x^2 = 3 + 1 - (1 + 2x + x^2) = 4 - (x+1)^2$. So if we set $x + 1 = 2\sin\theta$ we'd have $4 - (x+1)^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$. Since $x = 1 + 2\sin\theta$ we have $\mathrm{d}x = 2\cos\theta$ and we get dz

$$\begin{aligned} \int \left(3 - 2x - x^2\right)^{-3/2}\mathrm{d}x &= \int \left(4 - 4\sin^2\theta\right)^{-3/2} 2\cos\theta\,\mathrm{d}\theta \\ &= \frac{2}{4^{3/2}}\int \left(\cos^2\theta\right)^{-3/2}\cos\theta\,\mathrm{d}\theta \\ &= \frac{1}{4}\int \sec^2\theta\,\mathrm{d}\theta \end{aligned}$$

(5) (Final, 2008) Find the area inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: We need to compute the area between the curves $y = b\sqrt{1 - \frac{x^2}{a^2}}$ and $y = -b\sqrt{1 - \frac{x^2}{a^2}}$ for $-a \le x \le a$. We therefore substitute $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ to get

$$\begin{aligned} \text{Area} &= \int_{x=-a}^{x=a} 2b\sqrt{1 - \frac{x^2}{a^2}}\,\mathrm{d}x = 2b\int_{\theta=-\pi/2}^{\theta=\pi/2} \sqrt{1 - \sin^2\theta}a\cos\theta\,\mathrm{d}\theta \\ &= 2ab\int_{\theta=-\pi/2}^{\theta=\pi/2} \cos^2\theta\,\mathrm{d}\theta \\ &= ab\int_{\theta=-\pi/2}^{\theta=\pi/2} (1 + \cos(2\theta))\,\mathrm{d}\theta \\ &= ab\left[\theta + \frac{1}{4}\sin(4\theta)\right]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} = ab\left[\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) + \frac{1}{4}\sin(2\pi) - \frac{1}{4}\sin(-2\pi)\right] \\ &= \boxed{\pi ab}\,. \end{aligned}$$