Math 101 – SOLUTIONS TO WORKSHEET 5
INDEFINITE INTEGRALS

Theorem (Net change). *Suppose f' is continuous. Then $\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a)$.*

(1) (Net change theorem)

(a) A particle moves with velocity $v(t) = \pi\sin(\pi t)$. What is its displacement between the times $t = 0$ and $t = 2$?

Solution: Say the particle is at position $x(t)$. Then ("net change theorem")

$$\begin{aligned} x(2) - x(0) &= \int_{t=0}^{t=2} \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int_{t=0}^{t=2} v(t)\,\mathrm{d}t = \int_{t=0}^{t=2} \pi\sin(\pi t)\,\mathrm{d}t \\ &= [-\cos(\pi t)]_{t=0}^{t=2} = -\cos(2\pi) + \cos(0) = \boxed{0}. \end{aligned}$$

The particle is where it started.

(b) What is the total distance covered by the particle?

Solution: For $t \in [0,1]$ the particle is moving to the right, while for $t \in [1,2]$ it is moving to the left. The total distance covered is therefore

$$\begin{aligned} \int_{t=0}^{t=1} \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t + \int_{t=1}^{t=2}\left(-\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t &= [-\cos(\pi t)]_{t=0}^{t=1} - [-\cos(\pi t)]_{t=1}^{t=2} \\ &= (-(-1) - (-1)) - [-1 - (1)] \\ &= 4\,. \end{aligned}$$

In the alternative we would start with $\int_{t=0}^{t=2} |v(t)|\,\mathrm{d}t$ (distance travelled is the integral of the speed), but we'd immediately need to split into domains where $v(t) \geq 0$ and $v(t) \leq 0$, returning to the solution above.

(c) According to Newton's law of universal gravitation, the gravitational acceleration at distance r from a star of mass M is $a(r) = -\frac{GM}{r^2}$. The *gravitational potential* $\phi(r)$ is defined by $\phi'(r) = -a(r)$. What is the change in the gravitational potential between the surface of the Earth ($R_1 \approx 6,400$km) and geostational orbit ($R_2 \approx 42,000$km)? You may use $M_{\text{earth}} \approx 6\cdot 10^{24}$kg and $G \approx 6.7\cdot 10^{-11}\mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$.

Solution: $\phi(R_2) - \phi(R_1) = \int_{R_1}^{R_2} \phi'(r)\,\mathrm{d}r = -\int_{R_1}^{R_2} a(r)\,\mathrm{d}r = \int_{R_1}^{R_2} \frac{GM}{r^2}\,\mathrm{d}r = GM\left[-\frac{1}{r}\right]_{R_1}^{R_2} = \frac{GM}{R_1} - \frac{GM}{R_2}$. Plugging in the numerical values gives

$$\phi(R_2) - \phi(R_1) \approx 5.3\cdot 10^7\,\frac{\mathrm{m}^2}{\mathrm{sec}^2}\,.$$

(2) Find the indefinite integrals

(a) For $n \neq -1, \int x^n\,\mathrm{d}x =$

Solution: We know $\frac{\mathrm{d}}{\mathrm{d}x}x^{n+1} = (n+1)x^n$ so $\int x^n\,\mathrm{d}x = \boxed{\frac{1}{n+1}x^{n+1} + C}$.

(b) $\int\left(\frac{1}{2}x^{3/2} - e^{-x/3} + 7\right)\mathrm{d}x =$

Solution: We break the sum and then consider each piece separately. We note that $\left(x^{5/2}\right)' = \frac{5}{2}x^{3/2}$, $\left(e^{-x/3}\right)' = -\frac{1}{3}e^{-x/3}$ and get:

$$\begin{aligned} \int\left(\frac{1}{2}x^{3/2} - e^{-x/3} + 7\right)\mathrm{d}x &= \frac{1}{2}\int x^{3/2}\,\mathrm{d}x - \int e^{-x/3}\,\mathrm{d}x + \int 7\,\mathrm{d}x \\ &= \boxed{\frac{1}{2}\cdot\frac{2}{5}x^{5/2} + 3e^{-x/3} + 7x + C}. \end{aligned}$$

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(c) $\int_4^9 \left(x^{5/2} + e^{2x}\right) \mathrm{d}x =$

Solution: $\int \left(x^{5/2} + e^{2x}\right) = \frac{2}{7}x^{7/2} + \frac{1}{2}e^{2x} + C$ so

$$\begin{aligned}
\int_4^9 \left(x^{5/2} + e^{2x}\right) \mathrm{d}x &= \left[\frac{2}{7}x^{7/2} + \frac{1}{2}e^{2x}\right]_{x=4}^{x=9} \\
&= \frac{2}{7} \cdot 3^7 + \frac{1}{2}e^{18} - \frac{2}{7}2^7 - \frac{1}{2}e^8 \\
&= \boxed{\frac{2 \cdot 3^7 - 2^8}{7} + \frac{e^{18} - e^8}{2}}.
\end{aligned}$$

(d) $\int x \left(e^{x^2} + 1\right) \mathrm{d}x =$

Solution: $\int x \left(e^{x^2} + 1\right) \mathrm{d}x = \int xe^{x^2} \mathrm{d}x + \int x \, \mathrm{d}x$. For the first part we note that $\left(e^{x^2}\right)' = 2xe^x$ to get

$$\int x \left(e^{x^2} + 1\right) \mathrm{d}x = \boxed{\frac{e^{x^2} + x^2}{2} + C}.$$