23. Geometric Series (6/3/2017)

Goals:

(1) Limits of sequences: the squeeze theorem
(2) Geometric series
(3) Telescoping series and partial sums

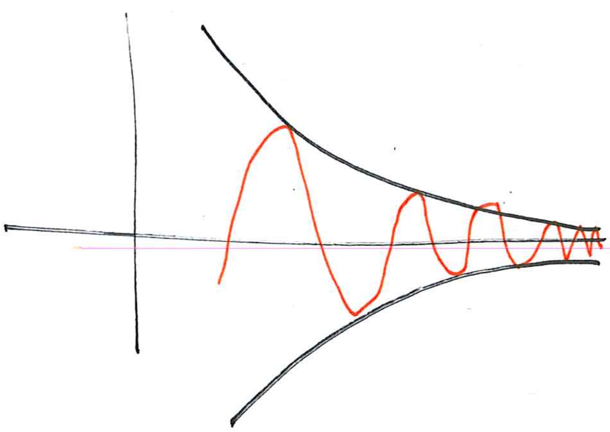
$\lim_{n\to\infty} \frac{n^3+5}{n^2-7} = \lim_{n\to\infty} \frac{n^3}{n^2} \cdot \frac{1+5/n^3}{1-7/n^2} = \left(\lim_{n\to\infty} n\right)\left(\lim_{n\to\infty} \frac{1+5/n^3}{1+7/n^2}\right) = \infty$

$\lim_{n\to\infty} \frac{100\cdot 2^{n+1}+5}{10\cdot 5^{2n-3}+2} = \lim_{n\to\infty} \frac{2^n}{5^{2n}} \cdot \frac{200+5/2^n}{\frac{10}{5^3}+2/5^{2n}} = \left(\lim_{n\to\infty} \left(\frac{2}{25}\right)^n\right)\left(\lim_{n\to\infty} \frac{200+5/2^n}{\frac{10}{5^3}+2/25^n}\right)$

$= 0 \cdot \text{const} = 0$

extract rates of growth/decay

Reminder: Squeeze thm applies to limits of sequences



In words: if $a_n \le b_n \le c_n$

and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} b_n = L$

Example: Consider $\lim_{n\to\infty} \frac{(-1)^n}{n^2}$. We have $-\frac{1}{n^2} \le \frac{(-1)^n}{n^2} \le \frac{1}{n^2}$

$-\frac{1}{n^2} \xrightarrow{n\to\infty} 0$, $\frac{1}{n^2} \xrightarrow{n\to\infty} 0$

By squeeze thm, $\lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0$

Math 101 – WORKSHEET 23
SERIES

1. Tool: Squeeze Theorem

(1) Determine if each sequence is convergent or divergent. If convergent, evaluate the limit.

(a) (Final 2013) $\left\{(-1)^n \sin\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$.

Have for all n.
$-\sin(\frac{1}{n}) \le (-1)^n \sin(\frac{1}{n}) \le \sin(\frac{1}{n})$. Now, $\lim_{n\to\infty} \sin(\frac{1}{n}) = \sin(\lim_{n\to\infty} \frac{1}{n}) = \sin(0) = 0$ (sin x is cts)

So $\lim_{n\to\infty} -\sin(\frac{1}{n}) = -0 = 0$. By squeeze thm, $\lim_{n\to\infty} (-1)^n \sin(\frac{1}{n}) = 0$ as well.

(b) (Final 2011) $\left\{\frac{\sin(n)}{\log(n)}\right\}_{n=2}^{\infty}$ (why do we have $n \ge 2$ here?)

For all n, $-1 \le \sin(n) \le 1$ so $-\frac{1}{\log n} \le \frac{\sin(n)}{\log(n)} \le +\frac{1}{\log(n)}$, $\lim_{n\to\infty} \log n = \infty$ so

$\lim_{n\to\infty} \pm\frac{1}{\log n} = \pm \lim_{n\to\infty} \frac{1}{\log n} = 0$, so By squeeze thm $\lim_{n\to\infty} \frac{\sin(n)}{\log(n)} = 0$ as well

(c) (Math 105 Final 2012) $a_n = 1 + \frac{n!\sin(n^3)}{(n+1)!}$.

Use $1 - \frac{1}{n+1} \le 1 + \frac{n!\sin(n^3)}{(n+1)!} \le 1 + \frac{1}{n+1}$

Date: 6/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Observe with Xeno: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$

Why? We saw

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^n}=1-\underset{\text{distance remaining to cross room}}{\frac{1}{2^n}}$$

So $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2^k}=\lim_{n\to\infty}\left(1-\frac{1}{2^n}\right)=1$

Or: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=1$

A formal sum $a_1+a_2+a_3+a_4+\cdots=\sum_{n=1}^{\infty}a_n$ is called a series.

A partial sum of the series is a finite sum $a_1+a_2+\cdots+a_N$ (also write $S_N=\sum_{n=1}^{N}a_n$)

Def: Say the series converges if the sums S_N tend to a limit. If $\lim_{n\to\infty}S_N=S$ write $\sum_{n=1}^{\infty}a_n=S$.

Example: Geometric series: $\sum_{n=0}^{\infty}a\cdot q^n=a+aq+aq^2+aq^3+\cdots$

Formula: The series $\sum_{n=0}^{\infty}q^n$ converges if and only if $|q|<1$ and then:

$$\sum_{n=0}^{\infty}q^n=\frac{1}{1-q}\ ,\qquad \sum_{n=0}^{\infty}aq^n=\frac{a}{1-q}$$

first term (a), ratio (q)

[Why? Because: $\sum_{n=0}^{N-1}aq^n=a\frac{1-q^N}{1-q}\xrightarrow[N\to\infty]{}a\frac{1-0}{1-q}=\frac{a}{1-q}$]

2. Skill 1: Geometric series and decimal expansions

(1) (Final 2013) Find the sum of the series $\sum_{n=2}^{\infty} \frac{3 \cdot 4^{n+1}}{8 \cdot 5^n}$. Simplify your answer.

This is a geometric series with ratio $\frac{4}{5}$, first term $\frac{3}{8} \cdot \frac{4^3}{5^2} = \frac{24}{25}$

It converges $(-1 < \frac{4}{5} < 1)$, and its sum is $\frac{24}{25} \cdot \frac{1}{1-4/5} = \frac{24}{5} \cdot \frac{1}{5-4} = \boxed{\frac{24}{5}}$

(2) Express each decimal expansion using a geometric series, sum the series, then simplify to obtain a rational number.

(a) 0.333333 . . . $= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10^4} + \dots = \frac{3/10}{1-1/10} = \frac{3}{9} = \frac{1}{3}$

ratio $= \frac{1}{10}$

(b) 0.5757575757 . . . $\frac{57}{100} + \frac{57}{10^4} + \frac{57}{10^6} + \dots = \sum_{n=1}^{\infty} \frac{57}{(100)^n}$

$= \frac{57/100}{1 - 1/100} = \frac{57}{99} = \frac{19}{33}$

(c) 0.6545454545454 . . . $= \frac{6}{10} + \frac{54}{1000} + \frac{54}{10^5} + \frac{54}{10^7} + \dots$

$= \frac{6}{10} + \frac{1}{10} \sum_{n=1}^{\infty} \frac{54}{(100)^n} =$

Example: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$ note: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$\frac{1}{2} + \frac{1}{6} = (1-\cancel{\tfrac{1}{2}}) + (\cancel{\tfrac{1}{2}} - \tfrac{1}{3}) = 1 - \frac{1}{3}$, $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} = (1-\cancel{\tfrac{1}{2}}) + (\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}) + (\cancel{\tfrac{1}{3}} - \tfrac{1}{4}) = 1 - \frac{1}{4}$

3. Skill 2: Telescoping series

(3) Write an expression for the partial sums, decide if the series converges, and if so determine the sum.

$\sum_{n=1}^{N} \frac{1}{n(n+1)} = (1-\cancel{\tfrac{1}{2}}) + (\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}) + \dots + (\cancel{\tfrac{1}{N-1}} - \cancel{\tfrac{1}{N}}) + (\cancel{\tfrac{1}{N}} - \tfrac{1}{N+1})$

$= 1 - \frac{1}{N+1} \xrightarrow[N\to\infty]{} 1$ so $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

(a) (Final 2015) $\sum_{n=3}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right)\right)$

cut off $\to$ $\sum_{n=3}^{N} \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right)\right) = \left(\cos\left(\frac{\pi}{3}\right) - \cancel{\cos\left(\frac{\pi}{4}\right)}\right) + \left(\cancel{\cos\left(\frac{\pi}{4}\right)} - \cancel{\cos\left(\frac{\pi}{5}\right)}\right) + \dots + \left(\cancel{\cos\left(\frac{\pi}{N}\right)} - \cos\left(\frac{\pi}{N+1}\right)\right)$

$= \cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{N+1}\right) \xrightarrow[N\to\infty]{} \frac{1}{2} - \cos(0) = \frac{1}{2} - 1 = -\frac{1}{2}$

So $\sum_{n=3}^{\infty} \left(\cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{\pi}{n+1}\right)\right) = -\frac{1}{2}$

(b) $\sum_{n=1}^{\infty} \left(n^2 - (n+1)^2\right)$

(c) $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$

(d) $\sum_{n=0}^{\infty} (\arctan(n) - \arctan(n+1))$