22. Sequences (3/3/2017)

Goals:

(1) Review Separaration of Variables
(2) Know what a sequence is
(3) Convert between formula and $\cdots$ notation
(4) Distinguish convergence from divergence
(5) Evaluate limits; arithmetic of limits
(6) Squeeze theorem

Last time: Separable DE. (1) Separate variables
(2) Integrate both sides (3) solve for y if possible
(4) Determine constant

Example (Final 2014). Find the solution to the equation $x\frac{dy}{dx} + y = y^2$ satisfying $y(1) = -1$.

(recall: unknown y represents a function of x)

① separate vars: $x\frac{dy}{dx} + y = y^2 \iff x\frac{dy}{dx} = y^2 - y \iff \frac{dy}{y^2-y} = \frac{dx}{x}$

② integrate: $\int \frac{dy}{y^2-y} = \int \frac{dx}{x}$: $\int \frac{dx}{x} = \log|x| + C$

$\int \frac{dy}{y^2-y} = \int dy\left(\frac{1}{y-1} - \frac{1}{y}\right) = \log|y-1| - \log|y| + C$

Aside: $\frac{1}{y^2-y} = \frac{1}{y(y-1)} = \frac{-1}{y} + \frac{1}{y-1} = \frac{1}{y-1} - \frac{1}{y}$

$\Rightarrow \log\left|\frac{y-1}{y}\right| = \log|x| + C$

③ solve for y: exponentiate both sides: $\left|\frac{y-1}{y}\right| = |x| \cdot e^C = C|x|$

$\Rightarrow$ by continuity, $\frac{y-1}{y} = Cx \Rightarrow 1 - \frac{1}{y} = Cx \Rightarrow \frac{1}{y} = 1 - Cx$

$\Rightarrow \boxed{y = \frac{1}{1-Cx}}$

Answer: $\boxed{y = \frac{1}{1-2x}}$

④ find C: Want $y(1) = -1$, i.e. $-1 = y(1) = \frac{1}{1-C}$ $\Rightarrow 1 - C = -1 \Rightarrow C = 2$

At home: check $e^{\log|x|+c} = |x| \cdot e^c$.

Sequences:

Ultimate goal: express a function (say e^x) by an infinite sum (e.g. $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$)

Today: start by considering infinite sequences

Sequence: function, domain = positive integers

Example: $\{1, 1, 1, 1, 1, \ldots\}$ in formula: $a_n = 1$

$a_1, a_2, a_3, a_4, \ldots$

Example: $\{3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, \ldots\}$

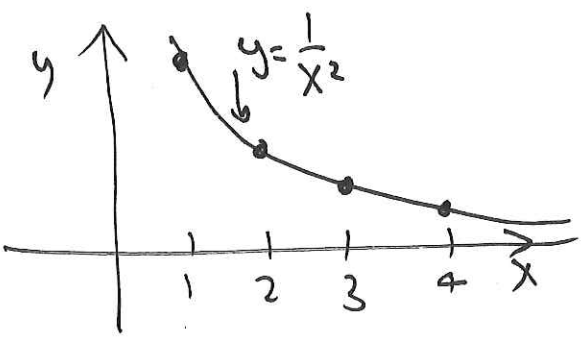
(sometimes no formula)

Worksheet 1

Def: The sequence $\{a_n\}_{n=1}^{\infty}$ tends to the limit L if it's eventually very close to L.

Notation: $\lim_{n\to\infty} a_n = L$.

Examples If $a_n = f(n)$, and $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} a_n = L$



$\lim_{x\to\infty} \frac{1}{x^2} = 0$

so

$\lim_{n\to\infty} \frac{1}{n^2} = 0$

Math 101 – WORKSHEET 22
SEQUENCES

1. Skill 1: expressions for sequences

(1) For each of the following sequences, write a formula for the general term

(a) $\{1, 2, 3, 4, 5, 6, \cdots\}$ $\to a_n = n$
$= \{n\}_{n=1}^{\infty}$

(b) $\left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \cdots\right\} = \left\{\frac{1}{n^2}\right\}_{n=1}^{\infty}$ or $a_n = \frac{1}{n^2}$

(c) $\{3, 7, 11, 15, 19, \cdots\}$ $a_n = 4n-1$ but we generally assume start at $n=1$
$= \cancel{\{3+4n\}}\ \{3+4n\}_{n=0}^{\infty}$

(d) $\left\{\frac{7}{9}, \frac{7}{27}, \frac{7}{81}, \frac{7}{243}, \frac{7}{729}, \frac{7}{3187}, \cdots\right\} = \left\{\frac{7}{3^{n+1}}\right\}_{n=1}^{\infty}$

(e) $\left\{\frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{1}{4}, \frac{5}{32}, \frac{3}{32}, \frac{7}{128}, \frac{1}{32}, \frac{9}{512}, \frac{5}{512} \cdots\right\} = \left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, \frac{}{1}\right.$

(f) $\{1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \cdots\} = \{(-1)^{n+1}\}_{n=1}^{\infty}$
$= \{\cos(\pi(n+1))\}_{n=1}^{\infty}$

(g) $\left\{0, \frac{3}{8}, \frac{2}{27}, \frac{5}{64}, \frac{4}{125}, \frac{7}{216}, \frac{6}{343}, \frac{9}{512}, \frac{8}{729}, \frac{11}{1000}, \cdots\right\}$

Date: 3/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

2. Skill 2: limits of sequences

(2) Determine if the sequences is convergent of divergent. If convergent, evaluate the limit.

(a) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

(b) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$

all formulas for limits still work

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+\lim\limits_{n\to\infty}\frac{1}{n}} = \frac{1}{1+0} = 1$$

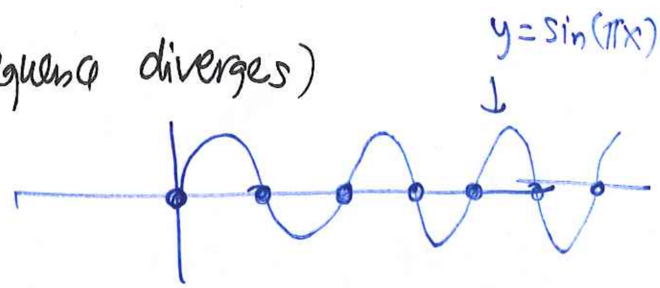
$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{1}{1} = 1$$

L'Hôpital

(c) $\{\sin(n)\}_{n=5}^{\infty}$

limit does not exist (say sequence diverges)

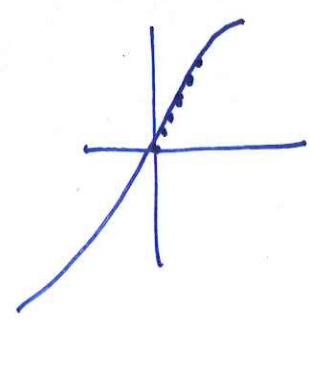
What about $\lim\limits_{n\to\infty} \sin(\pi n) = \lim\limits_{n\to\infty} 0 = 0$



(d) $\left\{\sin(\frac{1}{n})\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \sin\left(\lim_{n\to\infty}\frac{1}{n}\right) = \sin(0) = 0$$

sin(x) is cts



(3) Further problems

(a) Does $\lim_{n\to\infty} \frac{n}{\sqrt{n+1000}}$ exist? yes

$$\frac{n}{\sqrt{n+1000}} = \frac{\sqrt{n}\cdot\sqrt{n}}{\sqrt{n+1000}} = \frac{1}{\sqrt{1+\frac{1000}{\sqrt{n}}}}\cdot\sqrt{n} \xrightarrow[n\to\infty]{} 1\cdot\infty = \infty$$

no limit (yes in extended sense)

(b) $\lim_{n\to\infty} \frac{n}{2^n} = \lim_{x\to\infty} \frac{x}{2^x} = \lim_{x\to\infty} \frac{1}{(\log 2)\, 2^x} = 0$

l'Hôpital

(c) (Math 103 final, 2014) Consider the sequence $\{a_n\}_{n=1}^{\infty} = \left\{1, 0, \frac{1}{2}, 0, 0, \frac{1}{3}, 0, 0, 0, \frac{1}{4}, 0, 0, 0, 0, \frac{1}{5}, \cdots\right\}$. Decide whether $\lim_{n\to\infty} a_n = 0$.