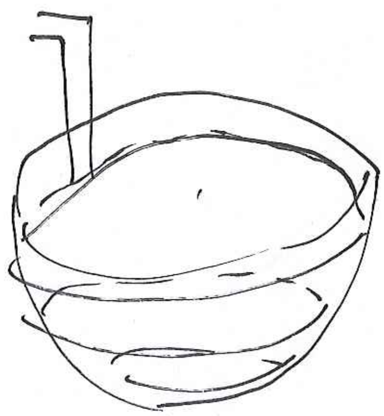
21. Separable DE (1/3/2017)

Goals:

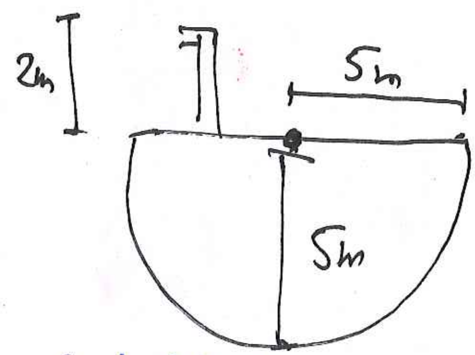
(1) Review CM & Work
(2) Know what a DE is
(3) Solve DE by separation of variables
    (a) Note: only one constant

Problem: Find the work required to drain a hemispherical bowl of water of radius 5m through a spout 2m above the surface of the water.

Solution: ① picture

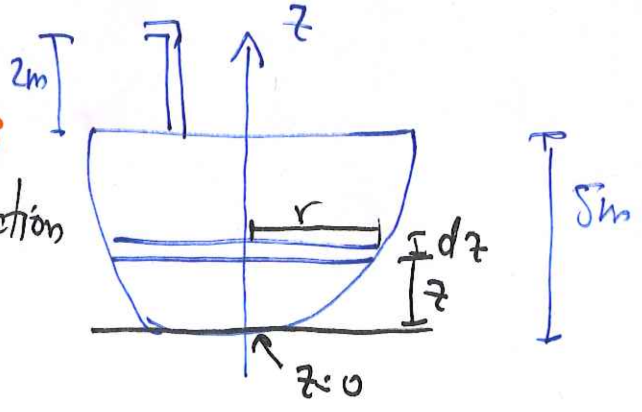


Cross-section



need to select slicing.
idea: take slice all of which will move same (vertical) distance
so want horizontal slices:

② name quantities



slice has circular cross-section thickness dz.

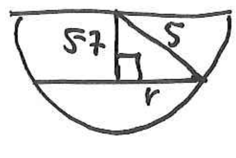
③ relations between them: volume of slice is $\pi r^2 dz$

weight of slice is $\pi \rho g r^2 dz$, ρ = density of water, g = acceleration of gravity work done on slice = (force) × (distance) =

$$dW = \pi \rho g \; r^2 dz \cdot (5 - z + 2)$$

Total work: $$W = \int_{z=0}^{z=5} dW = \int_{z=0}^{z=5} \pi \rho g r^2 dz (5 - z + 2)$$

Need relation between r, z.



Realise triangle between centre of hemisphere,
centre of slice
edge is right-angled

so $r^2 = 5^2 - (5-z)^2 = 10z - z^2$. ← key relation

④ Integral to compute

set $$W = \pi \rho g \int_0^5 (10z - z^2) \underbrace{(5 - z + 2)}_{7-z} dz$$

$$= \pi \rho g \int_0^5 (70z - 17z^2 + z^3) dz$$

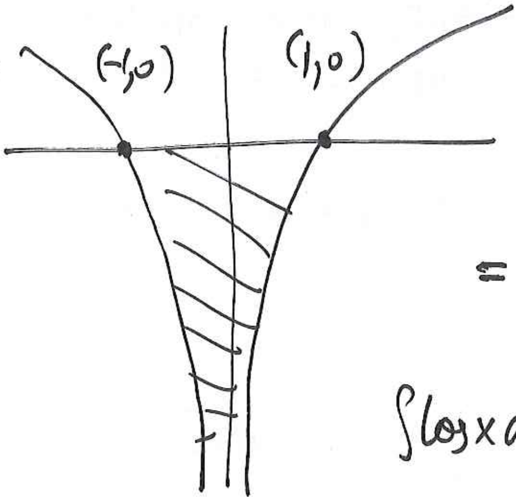
$$= \pi \rho g \left[35z^2 - \frac{17}{3} z^3 + \frac{z^4}{4} \right]_0^5 = \pi \cdot 9{,}800 \cdot \left(35 \cdot 25 - \frac{17 \cdot 125}{3} + \frac{625}{4} \right)$$

$\rho = 1000 \, \frac{kg}{m^3}$

$g = 9.8 \, \frac{m}{s^2}$

Problem: Find the CM of the region below the x axis, between branches of $\log|x|$

Solution:



Area: $-\int_{-1}^{1} \log|x|\,dx \overset{\text{symmetry}}{=} -2\int_{0}^{1} \log x\,dx =$

$$= \lim_{\varepsilon\to 0} \int_{\varepsilon}^{1} \log x\,dx$$

by parts:

$$\int \log x\,dx = x\log x - \int x\cdot\frac{1}{x}\,dx = x\log x - x + C$$

$$-2\int_0^1 \log x\,dx = -2\lim_{\varepsilon\to 0}\Big[x\log x - x\Big]_{\varepsilon}^{1} = -2\lim_{\varepsilon\to 0}\Big[-1-\varepsilon\log\varepsilon+\varepsilon\Big] =$$

$$= 2 - 2\lim_{\varepsilon\to 0}\varepsilon + 2\lim_{\varepsilon\to 0}\frac{\log\varepsilon}{1/\varepsilon} \underset{\text{l'Hôpital}}{=} 2 + 2\lim_{\varepsilon\to 0}\frac{1/\varepsilon}{-1/\varepsilon^2} = 2 - 2\lim_{\varepsilon\to 0}\varepsilon = 2.$$

$$\lim_{\varepsilon\to 0}\log\varepsilon = -\infty,\quad \lim_{\varepsilon\to 0}\frac{1}{\varepsilon} = \infty$$

$$\boxed{\text{Area} = 2}$$

By symmetry $\bar{x} = x_{CM} = \bar{X} = 0$ (region symmetric about y-axis)

$$-\bar{y} = \frac{1}{2(\text{Area})}\int_{-1}^{1}(\log x)^2\,dx \underset{\text{symmetry}}{=} \frac{1}{2}\int_0^1 \log^2 x\,dx$$

$$\int \log^2 x\,dx = x\log^2 x - \int x\cdot 2\log x\,\frac{1}{x}\,dx = x\log^2 x - 2\int \log x\,dx =$$

$$= x\log^2 x - 2(x\log x - x) = x\log^2 x - 2x\log x + 2x$$

So $\frac{1}{2}\int_0^1 \log^2 x\,dx = \frac{1}{2}\Big[2 - \lim_{\varepsilon\to 0}\big(\varepsilon\log^2\varepsilon - 2\varepsilon\log\varepsilon + 2\varepsilon\big)\Big] = 1 - 0 = 1.$

CM is at $(0, -1)$.

$$\boxed{\begin{array}{l}\bar{y} = \dfrac{1}{2\,\text{Area}}\displaystyle\int (f^2 - g^2)\,dx \\ \text{here } f = 0,\ g = \log x\end{array}}$$

A differential equation is an equation whose solution is a function and equation involved derivatives of that function

Math 101 – WORKSHEET 21
SEPARABLE DIFFERENTIAL EQUATIONS

1. What is a DE?

(1) Consider the differential equation $y' = 3y^2$

(a) For which values of C, D is $f(x) = Cx^D$ a solution to the equation?

Want: $CDx^{D-1} = 3C^2x^{2D}$ for all x.

to get equality of functions need $D = -1$ $(D-1=2D)$

either $C=0$, or $C = -\frac{1}{3}$ $\leftarrow CD = 3C^2$

so either ~~f(x) is~~ $y=0$ or $y = -\frac{1}{3}\frac{1}{x}$

(b) Suppose $f(x)$ is a solution. Show that $f(x-a)$ is also a solution for any a. What is the solution with $f(0) = 1$?

Different idea: use substitution:

~~equation~~ equivalence: $\frac{dy}{dx} = 3y^2 \Leftrightarrow \frac{dy}{3y^2} = dx \Rightarrow \int \frac{dy}{3y^2} = \int dx$

separate the variables

$$\Downarrow$$

$$-\frac{1}{3y} = x + C$$

(enough to have one constant)

Date: 1/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

so general solution: $y = -\frac{1}{3(x+C)}$

2. Separation of variables

(2) Solve the follwing equations using separation of variables

(a) $y' = x^3$ $\quad \Rightarrow \quad y = \frac{1}{4}x^4 + C$

or: $\frac{dy}{dx} = x^3 \quad \Rightarrow \quad dy = x^3 dx \quad \Rightarrow \quad y = \frac{x^4}{4} + C$

integrate

(b) $y' = 5y$

$\frac{dy}{dx} = 5y \quad \Rightarrow \quad \frac{dy}{y} = 5\,dx \quad \Rightarrow \quad \int \frac{dy}{y} = \int 5\,dx$

$\Rightarrow \log|y| = 5x + C \quad \Rightarrow \quad |y| = e^{5x+C} = e^C \cdot e^{5x}$

so $|y| = C \cdot e^{5x}$ for some C so either $y = Ce^{5x}$ or $y = -Ce^{5x}$

$C, -C$ both constant, so $\boxed{y = Ce^{5x}}$

(c) (Final, 2012) $y' = xy$, $y(0) = e$.