15. Partial Fractions, Numerical Integration (6/2/2017)

have regrades
IDs

Goals.

(1) Partial fractions:
   (a) Factorization of polynomials
   (b) Division of polynomials
   (c) Quadratic factors
(2) Numerial integration formulas to memorize

Consider $\frac{x^2}{(x+2)(2x-3)}$. Can we have A, B such that $x^2 = A(x+2) + B(2x-3)$?

(get this by clearing denominators from $\frac{x^2}{(x+2)(2x-3)} = \frac{A}{2x-3} + \frac{B}{x+2}$)

Problem: degree (numerator) $= 2 \geq 2 =$ degree (denominator)

Sample question: find partial fraction expansion of

$$\frac{x}{x^3-6x^2+11x-6}.$$

Math 101 – WORKSHEET 15
PARTIAL FRACTIONS, APPROXIMATE INTEGRATION

1. Partial fractions expansion

(1) Write the form of the partial fraction expansion:

$$\frac{x^2+3x+6}{x^3(2x+3)(x-1)^2} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{2x+3} + \frac{E}{(x-1)^2} + \boxed{\frac{F}{(x-1)}}$$

(no ~~quadratic~~ irreducible factors in denominator)

↑ "partial fraction"

(2) (Division)

(a) Factor $x^3 - 6x^2 + 11x - 6$

Try for a rational root: denominator divides 6
numerator divides 6

i.e a rational root would be one of ±1, ±2, ±3, ±6.

Here: $1^3 - 6\cdot1^2 + 11 - 6 = 0$ so $x-1 \mid x^3-6x^2+11x-6$. Divide.

$$\begin{array}{r} x^2-5x+6 \\ x-1 \,\big)\, x^3-6x^2+11x-6 \\ -\;x^3-x^2 \\ \hline -5x^2 \\ -5x^2+5x \\ \hline 6x \\ 6x-6 \\ \hline 0 \end{array}$$

$\Rightarrow$ so $x^3-6x^2+11x-6 = (x-1)(x^2-5x+6)$
$= (x-1)(x-2)(x-3)$

(b) Divide with remainder: $\frac{2x^3}{(x+2)(2x+3)} = \left(x - \frac{7}{2}\right) + \frac{\frac{37}{2}x+21}{2x^2+7x+6}$

↑ deg of numerator ≥ deg of denom

divide ↑ apply partial fractions here

$$\begin{array}{r} x - \frac{7}{2} \\ 2x^2+7x+6 \,\big)\, 2x^3 \\ -\;2x^3+7x^2+6x \\ \hline -7x^2-6x \\ -\;-7x^2-\frac{49}{2}x-21 \\ \hline +\frac{37}{2}x+21 \end{array}$$

Date: 6/2/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$4x^2-4x+5$ irred: $\Delta = 16-16\cdot 5<0$.

over irred quadratic have $Bx+C$

(3) Consider $\frac{8x-10}{4x^3-4x^2+5x} = \frac{8x-10}{x(4x^2-4x+5)} = \frac{A}{x} + \frac{Bx+C}{4x^2-4x+5}$

(a) Find A using method 2

$$\lim_{x\to 0} \frac{8x-10}{4x^2-4x+5} = -\frac{10}{5} = \boxed{-2.}$$

(b) Calculate $\frac{8x-10}{x(4x^2-4x+5)} - \frac{A}{x}$ to find B, C.

$$\frac{8x-10}{x(4x^2-4x+5)} - \frac{(-2)}{x} = \frac{(8x-10)+2(4x^2-4x+5)}{x(4x^2-4x+5)} =$$

$$= \frac{8x^2}{x(4x^2-4x+5)} = \frac{8x}{4x^2-4x+5}$$

so $B=8$

$C=0$

Conclusion: $\int \frac{8x-10}{4x^3-4x^2+5x}\,dx = \int\left(-\frac{2}{x} + \frac{8x}{4x^2-4x+5}\right)dx =$

← complete the square

$$= -2\log|x| + \int \frac{8x-4+4}{(2x-1)^2+4}\,dx =$$

break up 8x into: (multiple of $2x-1$)+(const)

$$= -2\log|x| + \int \frac{4(2x-1)\,dx}{(2x-1)^2+4} + \int \frac{4\,dx}{(2x-1)^2+4} =$$

$$= -2\log|x| + \log\left(4+(2x-1)^2\right) + \arctan\left(x-\tfrac{1}{2}\right) + C$$

Approximate integration

Rules: Say we are integrating $\int_a^b f(x)dx$.

Choose n, $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$, $\bar{x}_i = a + (i - \frac{1}{2})\Delta x$

Midpoint rule: $\int_a^b f(x)dx \approx \left(f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n)\right)\Delta x$

Trapezoid rule: $\int_a^b f(x)dx \approx \left(\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \dots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right)\Delta x$

Simpson's rule: (n even)

$$\int_a^b f(x)dx \approx \Big(\frac{1}{6}f(x_0) + \frac{1}{3}f(x_1) + \frac{2}{3}f(x_2) + \frac{4}{3}f(x_3) + \frac{2}{3}f(x_4) + \dots + \frac{4}{3}f(x_{n-1}) + \frac{1}{6}f(x_n)\Big)\Delta x.$$