9. Solids of revolution, Integration by parts (23/1/2017)

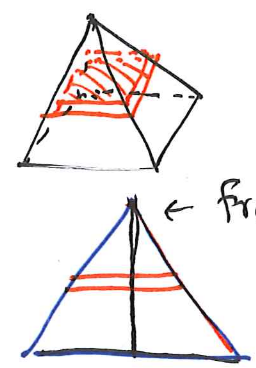
Goals.

(1) Solids of revolution
(2) Integration by parts
(3) Working with a toolkit.

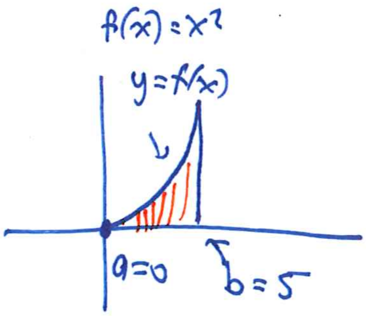
Last Time: Volume of the ball:

(1) Picture
(2) Slice
(3) Axes + geometry ⇒ volume element
(4) Write integral
(5) Evaluate integral

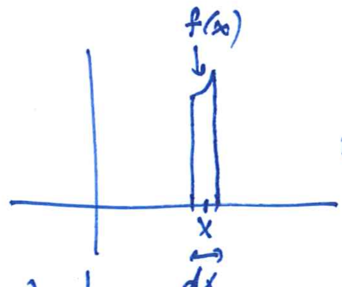
Example: pyramid



Problem: The area between the x-axis, the curve $y = x^2$ and the line $x = 5$ is rotated about the x-axis. What is the volume of the resulting region?



→ one slice



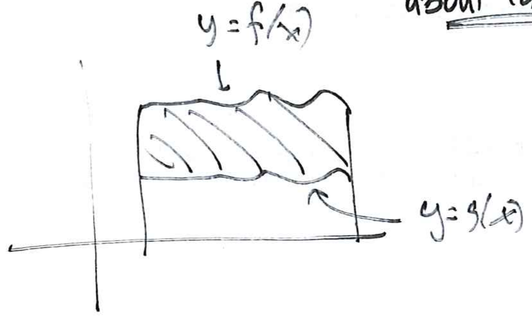
rotate

get a thin wafer, radius $f(x)$, thickness dx, volume $\pi(f(x))^2dx$

So the total volume of the region is $\boxed{\pi\int_a^b (f(x))^2dx.}$

Here: $f(x)=x^2$, $a=0$, $b=5$, volume $= \pi\int_{x=0}^{x=5}(x^2)^2dx = \frac{\pi}{5}\left[x^5\right]_0^5 = 625\pi.$

From this: If we rotate the area between the graphs of $f(x), g(x)$ about the x-axis (say $f(x) \geq g(x) \geq 0$) the volume is



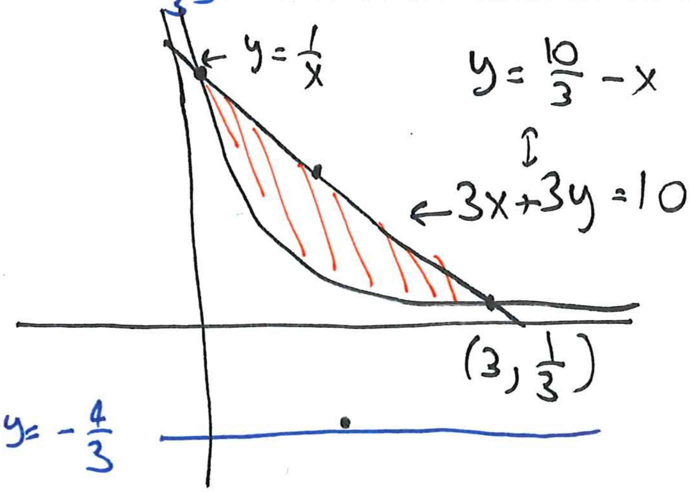
get volume $\pi \int_a^b \left((f(x))^2 - (g(x))^2 \right) dx$

What if axis is not x-axis? replace f, g in formula with distances to axis.

Math 101 – WORKSHEET 9
SOLIDS OF REVOLUTION, INTEGRATION BY PARTS

(1) Solids of revolution

(a) (Final 2014, variant) Find the volume of the solid generated by rotating the finite region bounded by $y = \frac{1}{x}$ and $3x + 3y = 10$ about the line $y = -\frac{4}{3}$. It will be useful to sketch the region first.



intersection: where

$$3x + \frac{3}{x} = 10$$

$$\Downarrow$$

$$3x^2 - 10x + 3 = 0$$

$$x = \frac{10 \pm \sqrt{100-36}}{6} = \frac{10 \pm 8}{6} = 3, \frac{1}{3}$$

By formula, the volume of the region is

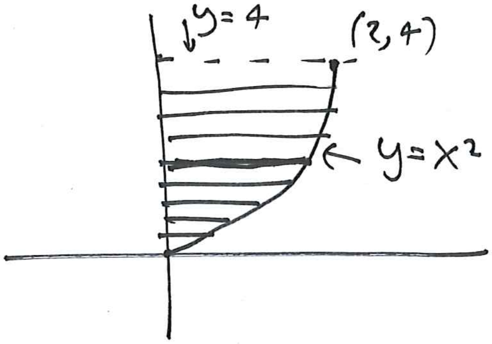
$$\pi \int_{1/3}^{3} \left(\left(\frac{10}{3} - x + \frac{4}{3} \right)^2 - \left(\frac{1}{x} + \frac{4}{3} \right)^2 \right) dx$$

distance to axis of rotation

$$= \pi \int_{1/3}^{3} \left(\frac{196}{9} - \frac{28}{3}x + x^2 - \frac{1}{x^2} - \frac{8}{3x} - \frac{16}{9} \right) dx = \ldots$$

Date: 23/1/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(b) The area between the y-axis, the curve $y = x^2$ and the line $y = 4$ is rotated about the y-axis. What is the volume of the resulting region?



By formula, the volume is

$$\pi \int_{y=0}^{y=4} x^2\,dy \underset{x^2=y}{=} \pi \int_0^4 y\,dy = \pi \left[\frac{y^2}{2}\right]_0^4 = 8\pi.$$

$$\text{volume} = \pi \int_*^* \left(\begin{smallmatrix}\text{distance}\\ \text{to axis}\end{smallmatrix}\right)^2 \cdot d(\text{variable})$$

Alternative, this is same as "area between x-axis, $x=y^2$, line $x=4$ rotated about x-axis".

Reminder: substitution was about looking at integrand, dividing into $f(g(x)) \cdot g'(x)dx$

Product rule: $(uv)' = u'v + \cancel{}\, uv'$

so $\cancel{}$ $uv' = (uv)' - u'v$

$$\Downarrow$$

$$\int uv'dx = uv - \int u'v\,dx$$

Conclusion: Write the integrand as $f \cdot g'$
replace problem with one involving $f' \cdot g$

(2) Integrate by parts

(a) $\int xe^x \, \mathrm{d}x = x \cdot e^x - \int e^x \, dx = x \cdot e^x - e^x + C$

$u = x, \quad dv = e^x dx$

$du = dx, \quad v = e^x$

(b) (Final, 2014) $\int x \log x \, \mathrm{d}x$