MATH 101: INTEGRATING THE SECANT FUNCTION

LIOR SILBERMAN

In this note we'll compute $\int \tan x \, \mathrm{d}x$ and $\int \sec x \, \mathrm{d}x$ by combining some ideas from class.

Remark. You are required to be able to compute $\int \tan x \, \mathrm{d}x$ during exams (either by memorizing the answer or by applying any valid method of integration). On the other hand if a problem requires $\int \sec x \, \mathrm{d}x$ then the answer to this bit will be provided.

1. $\int \tan x \, \mathrm{d}x$

Write the integral as $\int \frac{\sin x}{\cos x} \mathrm{d}x$ and we immediately see that if we set $u = \cos x$ we have $\mathrm{d}u = -\sin x \, \mathrm{d}x$ which is basically the denominator. We therefore have

$$\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \mathrm{d}x = \int \frac{-\mathrm{d}u}{u} = -\log|u| + C = -\log|\sin x| + C \,.$$

2. $\int \sec x \, \mathrm{d}x$

Step 1: Trig integral. Let's write our integral as $\int (\cos x)^{-1} \mathrm{d}x$ (note that some people use $\cos^{-1} x$ to denote $\arccos x$, so we avoid that notation). We note that we need to integrate an *odd power of cosine.* In class we learned a rule for such trig integrals: take one power of $\cos x$ into $\cos x \, \mathrm{d}x$, and convert the rest of the cosines into sines. Here we have:

$$\int \frac{\mathrm{d}x}{\cos x} = \int \frac{\cos x \, \mathrm{d}x}{\cos^2 x} = \int \frac{\cos x \, \mathrm{d}x}{1 - \sin^2 x}$$

Step 2: u-substitution. According to the rule from above, we should substitute $u = \sin x$, $\mathrm{d}u = \cos x \, \mathrm{d}x$. This will give as

$$\int \frac{\mathrm{d}x}{\cos x} = \int \frac{\mathrm{d}u}{1 - u^2} \,.$$

Step 3: Partial fractions. We recognize $\frac{1}{1-u^2}$ are a rational function, which we could attack using partial fractions. Since the degree of the numerator is less than that of the denominator, we don't need to divide and can proceed to factoring the denominator. We clearly have $1 - u^2 = (1-u)(1+u)$ with roots at $u = \pm 1$. We have

$$\begin{aligned} \lim_{u \to 1} \frac{1}{(1+u)} &= \frac{1}{2} \\ \lim_{u \to -1} \frac{1}{(1-u)} &= \frac{1}{2} \end{aligned}$$

and hence

$$\frac{1}{1-u^2} = \frac{1}{2} \left(\frac{1}{1-u} + \frac{1}{1+u} \right) \,.$$

Step 4: Integration. We know that $\int \frac{\mathrm{d}u}{1+u} = \log|1+u| + C$. Also $\int \frac{\mathrm{d}u}{1-u} = -\int \frac{\mathrm{d}u}{u-1} = -\log|u-1|$. It follows that

$$\int \frac{\mathrm{d}u}{1-u^2} = \frac{1}{2} \log|1+u| - \frac{1}{2} \log|1-u| + C = \frac{1}{2} \log\left| \frac{1+u}{1-u} \right| + C \,.$$

This note is specifically excluded from the terms of UBC Policy 81.

Step 5: Back-substitution. We want an answer in terms of x. But $u = \sin x$ so the answer is

$$\int \frac{\mathrm{d}u}{1-u^2} = \frac{1}{2}\log\frac{1+\sin x}{1-\sin x} + C$$

(we can remove the absolute value sign since $1+\sin x, 1-\sin x$ are both non-negative).