Lior Silberman's Math 539: Problem Set 3 (due 30/3/2016)

Convergence Dirichlet Series

1. (Convergence of Dirichlet series) Let $D(s) = \sum_{n\geq 1} a_n n^{-s}$ be a formal Dirichlet series. We will study the convergence of this series as s varies in $\mathbb{C}$.
   (a) Suppose that $D(s)$ converges absolutely at some $s_0 = \sigma_0 + it$. Show that $D(s)$ converges uniformly absolutely in the closed half-plane $\Re(s) = \sigma \geq \sigma_0$.
   (b) Conclude that there is an *abcissa of absolute convergence* $\sigma_{\mathrm{ac}} \in [-\infty, +\infty]$ such that one of the following holds: (1) ($\sigma_{\mathrm{ac}} = \infty$) $D(s)$ does not converge absolutely for any $s \in \mathbb{C}$; (2) ($\sigma_{\mathrm{ac}} \in (-\infty, +\infty)$)$D(s)$ converges absolutely exactly in the half-plane $\sigma > \sigma_{\mathrm{ac}}$ or $\sigma \geq \sigma_{\mathrm{ac}}$; (3) ($\sigma_{\mathrm{ac}} = -\infty$) $D(s)$ converges absolutely in $\mathbb{C}$. In cases (2),(3) the convergence is uniform in any half-plane whose closure is a proper subset of the domain of convergence.
   (c) Suppose that $D(s)$ converges at some s_0. Show that $D(s)$ converges in the open half-plane $\sigma > \sigma_0$, locally uniformly in every half-plane of the form $\sigma \geq \sigma_1 > \sigma_0$, and that $D(s)$ converges absolutely in the half-plane $\sigma > \sigma_0 + 1$.
   (d) Conclude that there is an *absicssa of convergence* $\sigma_{\mathrm{c}} \in [-\infty, \infty]$ such that on of the following holds: (1) ($\sigma_{\mathrm{c}} = \infty$) $D(s)$ does not converge for any $s \in \mathbb{C}$; (2) ($\sigma_{\mathrm{c}} \in (-\infty, +\infty)$)$D(s)$ converges in the open half-plane $\sigma > \sigma_{\mathrm{c}}$ and diverges in the open half-plane $\sigma < \sigma_{\mathrm{c}}$; the convergence is locally uniform in any half-plane $\sigma \geq \sigma_1 > \sigma_{\mathrm{c}}$ (3) ($\sigma_{\mathrm{ac}} = -\infty$) $D(s)$ converges absolutely in $\mathbb{C}$. In cases (2) the convergence is uniform in any half-plane. Furthermore, σ_{c} and σ_{ac} are either both $-\infty$, both $+\infty$, or both finite, and in the latter case $\sigma_{\mathrm{c}} \leq \sigma_{\mathrm{ac}} \leq \sigma_{\mathrm{c}} + 1$.

2. Let $D(s)$ have abcissa of absolute convergence σ_{ac}.
   (a) Suppose $\sigma_{\mathrm{ac}} \geq 0$. Show that $\sum_{n\leq x} |a_n| \ll_\varepsilon x^{\sigma_{\mathrm{ac}}+\varepsilon}$.
   (b) Suppose $\sigma_{\mathrm{ac}} < 1$. Show that $\sum_{n>x} |a_n| n^{-1} \ll_\varepsilon x^{\sigma_{\mathrm{ac}}+\varepsilon}$.

3. (Convergence of sums and products) Let $D_1(s) = \sum_{n\geq 1} a_n n^{-s}$ and $D_2(s) = \sum_{n\geq 1} b_n n^{-s}$, and let $(D_1 + D_2)(s) = \sum_{n\geq 1} (a_n + b_n) n^{-s}$, $(D_1 \cdot D_2)(s) = \sum_{n\geq 1} c_n n^{-s}$ where $c = a * b$ is the Dirichlet convolution.
   (a) Show that the domain of absolute convergence of $D_1 + D_2$ and $D_1 D_2$ is at least the intersection of the domains of absolute convergence of D_1, D_2.
   (**b) (Mertens) Suppose that D_1, D_2 have abcissa of convergence σ_{c}. Show that $D_1 D_2$ has abcissa of convergence at most $\sigma_c + \frac{1}{2}$.

4. (Uniqueness of Dirichlet series) Suppose that $D(s) = \sum_{n\geq 1} a_n n^{-s}$ converges somewhere.
   (a) Suppose that $a_n = 0$ if $n < N$ and $a_N \neq 0$. Show that $\lim_{\Re(s)\to\infty} N^s D(s) = a_N$.
   (b) Suppose that $D_2(s) = \sum_{n\geq 1} b_n n^{-s}$ also converges somewhere, and that $D(s_k) = D_2(s_k)$ for $\{s_k\}$ in the common domain of convergence such that $\lim_{k\to\infty} \Re(s_k) = \infty$. Show that $a_n = b_n$ for all n.

5. (Landau's Theorem; proof due to K. Kedlaya) Let $D(s) = \sum_{n\geq 1} a_n n^{-s}$ have non-negative coefficients.
   (a) Show that $\sigma_c = \sigma_{ac}$ for this series.
   (b) Suppose that $D(s)$ extends to a holomorphic function in a small ball $|s - \sigma_c| < \varepsilon$. Show that if $s < \sigma_c < \sigma$ and s, σ are close enough to σ_c then s is in the domain of convergence of the Taylor expansion of D at σ.
   (c) Using that $D^{(k)}(\sigma) = \sum_{n=1}^{\infty} a_n(-\log n)^k n^{-\sigma}$, write $D(s)$ as the sum of a two-variable series with positive terms.
   (d) Changing the order of summation, show that $D(s)$ converges at s, a contradiction to the definition of σ_c.
   (e) Obtain *Landau's Theorem*: if $D(s)$ has positive coefficients, has abcissa of convergence σ_c, and agrees with a holomorphic function in some punctured neighbourhood of σ_c then the singularity at $s = \sigma_c$ is not removable.

Hadamard's Three-Line Theorem and the convexity bound

6. Let f be continuous in the strip $a \leq \Re(z) \leq b$, holomorphic in the interior of the strip. Suppose that $|f(x+iy)| = e^{o(y^2)}$ as $y \to \infty$ in the strip.
   (a) (Simple version) Suppose that $M_0 = \sup\{|f(z)| : \Re(z) = a\}$ and $M_1 = \sup\{|f(z)| : \Re(z) = b\}$ as finite. Show that for $x_t = (1-t)a + tb$ ($t \in [0,1]$) we have
   $$|f(x_t + iy)| \leq M_0^{1-t} M_1^t \,.$$
   (Hint: apply the maximum principle to the function
   $$g_\varepsilon(z) = f(z) M_0^{\frac{z-b}{b-a}} M_1^{\frac{z-a}{a-b}} e^{-\varepsilon z^2}).$$
   (b) (f growing) Suppose now that $|f(a+iy)| \ll |y|^{m_0}$, $|f(b+iy)| \ll |y|^{m_1}$ where $m_0, m_1 \geq 0$. Show that
   $$|f(x_t + iy)| \ll |y|^{m_t}$$
   where $m_t = (1-t)m_0 + tm_1$. (Hint: multiply and divide by functions of the form $\Gamma(\alpha z + \beta)$ for appropriate α, β).

7. (Application to functional analysis) Let (Ω, μ) be a measure space and $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Show that for $f \in L^p(\mu)$, $g \in L^q(\mu)$ we have *Hölder's inequality*,
   $$\int |fg| \, \mathrm{d}\mu \leq \|f\|_p \|g\|_q \,.$$
   (Hint: Consider $F(z) = \int_\Omega |f(\omega)|^{pz} |g(\omega)|^{q(1-z)} \, \mathrm{d}\mu(\omega)$ on the strip $0 \leq x \leq 1$).

Counting with Dirichlet Series

The following problems apply Theorem 113 of the notes.

8. PS1, problems 3, 4.
   (a) Estimate (with error terms) $\sum_{n\le x}\phi(n)$, $\sum_{n\le x}\frac{\phi(n)}{n}$, $\sum_{n\le x}\frac{\phi(n)}{n^2}$.
   (b) Show $\frac{1}{x}\sum_{n\le x}d_k(n) = P_k(\log x) + O(x^{-\frac{1}{k}})$ where P_k is a polynomial of degree $k-1$.
   (c) Show that $\sum_{n\le x}\sigma_\alpha(n) = Cx^{1+\alpha} + O(x^\beta)$ for some $\beta < \alpha$.

9. PS1, problem 8,9.
   (a) Let $a_p \in \mathbb{C}$ satisfy $|a_p| \le p^{-\sigma}$ and let $f(n) \overset{\text{def}}{=} \prod_{p|n}(1+a_p)$. Show that $\sum_{n\le x} f(n) = cx + O(x^{1-\sigma})$ where $c = \prod_p \left(1+\frac{a_p}{p}\right)$.
   (b) $\mathcal{A}_n$ denote a set of representative for the isomorphism classes of abelian groups of order n, $A_n = \#\mathcal{A}_n$ the number of isomorphism classes. Show that $\sum_{n\le x} A_n = cx + O\left(x^{1/2}\right)$ where $c = \prod_{k=2}^{\infty}\zeta(k)$.

10. (2014 Miklós Schweitzer competition) For $n \ge 2$ let $f(n)$ be the number of representations of n as a product of an ordered tuple of integers at least 2 and set $f(1) = 1$. Show that

$$\sum_{n\le x} f(n) = Cx^\alpha + \text{lower order}\ ,$$

where $\alpha > 1$ satisfies $\zeta(\alpha) = 2$.