Last time: ~~Group~~ Conjugation

G acts on G by ${}^g x = gxg^{-1}$.

$Z_G(x) = \{g \mid gx = xg\}$. Bijection $G/Z_G(x) \leftrightarrow$ conjugacy class of x

Theorem: ("Class equation") Let G be finite. Then

$$\#G = \#Z(G) + \sum_x [G : Z_G(x)]$$

where the sum is over representatives for the non-central conjugacy classes.

Pf: G is the (disjoint) union of its conjugacy classes so its size is the sum of their sizes.
The size of the class of x is $[G : Z_G(x)] = \# G/Z_G(x)$.

Conjugacy of subgroups

Def: For $g \in G$, $H < G$ set ${}^g H = gHg^{-1} = \{ghg^{-1} \mid h \in H\}$.

Lemma: This is a G-action.

Pf: $\gamma_g(x) = gxg^{-1}$ is a homomorphism $G \to G$ (an automorphism, in fact) and ${}^g H =$ image of H by γ_g. So gHg^{-1} is a subgp.
That this is an action is true in general (induced action from X to subsets of X)

Example: The class of H is $\{H\}$ iff H is normal.

Lemma: conjugacy of subgps is an equivalence relation

Pf: ${}^e H = eHe^{-1} = H$, ${}^{g^{-1}}({}^g H) = H$ so ${}^g H$ is conj. to H (symmetry)
${}^k({}^g H) = {}^{kg} H$ so transitivity

Lemma: There is a bijection between $\left\{\begin{smallmatrix}\text{subgps} \\ \text{conj. to } H\end{smallmatrix}\right\}$ and $G/N_G(H)$

Pf: map $\underset{G/N_G(H)}{gN_G(H)} \mapsto gHg^{-1}$!

well defined: if $gN_G(H) = g'N_G(H)$ then $g' = gn$ for $n \in N_G(H)$

then $g'Hg'^{-1} = (gn)H(gn)^{-1} = g(nHn^{-1})g^{-1} \underset{n\in N_G(H)}{=} gHg^{-1}$

$(N_G(H) = \{n \in G \mid nHn^{-1} = H\})$

Orbits, stabilizers

Let G act on X.

Def: Say $x, y \in X$ are in the same orbit if $y = g\cdot x$ for some $g \in G$

(orbit of x is $\{g\cdot x \mid g \in G\}$)

Lemma: This is an equivalence relation

Pf: (1) $e\cdot x = x$ (reflexivity) (2) if $y = g\cdot x$ then $g^{-1}y = g^{-1}(gx)$ $= (g^{-1}g)x = ex = x$

so $x = g^{-1}y$

(3) if $y = g\cdot x$, $z = h\cdot y$ then $z = h(gx) = (hg)\cdot x$

(transitivity)

~~Lemma: The orbit~~ Def: The stabilizer of x is $\mathrm{Stab}_G(x) = \{g \in G \mid g\cdot x = x\}$

Lemma: $\mathrm{Stab}_G(x)$ is a subgp and we have bijection

$$\{\text{orbit of } x\} \leftrightarrow G/\mathrm{Stab}_G(x)$$

Pf: (1) $e\cdot x = x$ so $e \in \mathrm{Stab}_G(x)$, if $gx = x$, $hx = x$ then $h^{-1}x = h^{-1}(hx) = x$

and $(gh)x = g(hx) = gx = x$ so $gh, h^{-1} \in \mathrm{Stab}_G(x)$

(2) map $g\,\mathrm{Stab}_G(x) \mapsto g\cdot x$

well-def: if $g\cdot \mathrm{Stab}_G(x) = g'\cdot \mathrm{Stab}_G(x)$ then $g' = gs$, $s\cdot x = x$.

Then $g'\cdot x = (gs)x = g(sx) = gx$.

surjection: if $y \in$ orbit of x then for some $g \in G$, $y = gx$

injection: if $gx = g'x$ then $g^{-1}g'\cdot x = g^{-1}gx = x$ so $g^{-1}g' \in \mathrm{Stab}_G(x)$

so $g\,\mathrm{Stab}_G(x) = g'\,\mathrm{Stab}_G(x)$

Note: G acts on G/H, by $g\cdot(xH) = (gx)H$ (induced action on subsets from regular action)

Ex: The map $\varphi : G/\mathrm{Stab}_G(x) \to X$

$g\,\mathrm{Stab}_G(x) \mapsto gx$

is a map of G-sets: $\varphi(g\cdot C) = g\varphi(C)$ for all $g \in G$, $C \in G/\mathrm{Stab}_G(x)$

Prop (Orbit-Stabiliser thm): if x, y are same orbit, $\mathrm{Stab}_G(x)$, $\mathrm{Stab}_G(y)$ are conjugate, and

$$\# X = \sum_{\underbrace{\{x\}}_{\text{orbit of } x} \in \underbrace{G\backslash X}_{\text{set of orbits of } G \text{ in } X}} [G : \mathrm{Stab}_G(x)]$$

Pf: same: X is the disjoint union of the orbits

Example: Suppose $\# G = p^k$ and X is finite.

By Lagrange's thm, $[G : \mathrm{Stab}_G(x)] \mid p^k$ so they are powers of p

so every orbit either has size 1 of has size is divisible by p

Call $x \in X$ fixed by G if $\mathrm{Stab}_G(x) = G$. Write $\mathrm{Fix}(X)$ for the set of fixed points.

Conclusion: $\# X = \# \mathrm{Fix}(X)$ + terms divisible by p

so

$$\# X \equiv \# \mathrm{Fix}(X) \mod p$$

Summary: Generalized phenomena in conjugation to actions in general: (1) orbit $\leftrightarrow$ $G/\mathrm{Stab}_G(x)$

(2) orbit-stabilizer thm

(3) Counting arguments are useful

Examples of group actions

(1) $GL_n(\mathbb{R})$ acts on $\mathbb{R}^n$: ($g \cdot \underline{v}$ = matrix-vector mult.)

orbit of $\underline{0}$ is $\{\underline{0}\}$,

other orbit is $\mathbb{R}^n \setminus \{\underline{0}\}$: let $\underline{u}_1, \underline{v}_1 \in \mathbb{R}^n \setminus \{0\}$. Want g s.t. $g\underline{u}_1 = \underline{v}_1$.

complete $\underline{u}_1$ to a basis $\{\underline{u}_1, \dots, \underline{u}_n\}$. Let $g: \mathbb{R}^n \to \mathbb{R}^n$ be the linear map

" $\underline{v}_1$ " " $\{\underline{v}_1, \dots, \underline{v}_n\}$ s.t. $g\underline{u}_i = \underline{v}_i$.

g is invertible because the map h s.t. $h\underline{v}_i = \underline{u}_i$ is inverse to it.

For $O(n)$, orbits are $\leftrightarrow$ lengths of vectors