Lecture 10

Last time: $N \overset{G}{<}$ normal if $gNg^{-1} = N$ for all $g \in G$

$\iff gN = Ng$ for all $g \in G$

$\iff$ setting $gN \cdot hN = ghN$ gives a group structure on G/N.

Call this group "G mod N", it came with a "quotient homomorphism" $q : G \to G/N$, $q(g) = gN$, q is surjective, $\mathrm{Ker}(q) = N$.

Also have congruence relation $g \equiv h\ (N) \iff \exists n \in N : g = hn$

$\iff h^{-1}g \in N$.

Remark: ~~Set~~ Fix $X \subset G$. Suppose we want to "kill" elements of X (set them to e)

then any element of $\langle X \rangle$ "dies" as well.

Also, if $x \equiv e$, then $gxg^{-1} \equiv e$ as well

conclusion: to "kill" X, must "kill" $\langle X \rangle^N$, this is sufficient by considering $G/\langle X \rangle^N$.

<u>Today</u>: (1) Technical tools "isomorphism theorems".
(2) Simple groups, A_n.

<u>Thm</u>: ("1st Isomorphism Theorem"): Let $f: G \to H$ be a hom, let $K = \mathrm{Ker}(f)$. Then f induces an isom

$$\bar{f}: G/K \xrightarrow{\sim} \mathrm{Im}(f)$$

pictorially:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "q"'] & H \\ & G/K \arrow[u, dashed, "\bar{f}"'] \end{tikzcd}$$

there a unique $\bar{f}: G/K \to H$ s.t. $\bar{f} \circ q = f$.

<u>Pf</u>: Define $\bar{f}(gK) := f(g)$. "choose some $x \in gK$, take $f(x)$"

("f is const on cosets, set $\bar{f}(gK) = f(gK)$")

<u>Well-defined</u>: for any $k \in K$, $f(gk) = f(g)f(k) \underset{k \in \mathrm{Ker}(f)}{=} f(g)e = f(g)$

<u>Hom</u>: $\bar{f}(gK \cdot hK) \underset{\text{def of } \cdot \text{ in } G/K}{=} \bar{f}(gh \cdot K) \underset{\text{def of } \bar{f}}{=} f(gh) \underset{f \in \mathrm{Hom}(G,H)}{=} f(g)f(h) \underset{\text{def of } \bar{f}}{=} \bar{f}(gK)\bar{f}(hK)$

<u>Injective</u>: $gK \in \mathrm{Ker}\,\bar{f} \iff \bar{f}(gK) = e \iff f(g) = e \iff g \in \mathrm{Ker}(f) \iff gK = e_{G/K}$.

<u>Surjective</u>: Let $h \in \mathrm{Im}(f)$. Then $h = f(g)$ for some $g \in G$.
Then $h = \bar{f}(gK) \in \mathrm{Im}(\bar{f})$

Thm: (2nd Isom Thm) let $N, H < G$ with N normal.

Then $HN = NH$ is a subgp, $N \cap H$ is normal in H, and the inclusion $\iota : H \hookrightarrow HN$ induces an isomorphism

$$H/_{N \cap H} \cong HN/_N$$

("add N to H, then kill off N. This is the same as killing off $N \cap H$ in H".)

Pf: (It is proved in PS 5 that HN is a subgroup of G)

let $q : HN \to HN/_N$ be the quotient map

let $f : H \to HN/N$ be the composition $f = q \circ \iota$

$$H \xrightarrow{\iota} HN \xrightarrow{q} HN/_N$$

concretely: $f(h) = hN \in (HN)/_N$

$\ker(f) = \{ \underset{\in H}{h} \mid hN = \underset{\text{id of } HN/_N}{N} \} = \{ h \in H \mid h \in N \} = H \cap N$

$\operatorname{Im}(f) = HN/_N$: for $hn \in HN$, we have $hn \cdot N = h(nN) = hN = f(h)$

By 1st isom thm, $H/\ker(f) \cong \operatorname{Im}(f)$ i.e. $H/_{H \cap N} \cong HN/_N$.

Thm: (3rd Isom thm) let $K < N < G$ be subgps, with $K, N \trianglelefteq G$.

Then N/K is normal in G/K and $(G/K)/(N/K) \cong G/N$.

Pf: (sketch) compose quotient maps $G \to G/K \to (G/K)/(N/K)$ call composition f, apply 1st isom thm

Simple Groups

<u>Motivation</u>: if G has normal subgp N, tempting to study N, G/N, and the ways to put them together

<u>Def</u>: G is <u>simple</u> if its only normal subgps are G, $\{e\}$

<u>Example</u>: C_p, p prime is simple (no <u>subgps</u> other than C_p, $\{e\}$)

(<u>Warning</u>: "simple group" almost always means "<u>non-abelian</u> simple group")

<u>Thm</u>: A_n is simple if $n \geq 5$.

<u>Fact</u>: There is a <u>classification</u> of all finite simple groups (starting point: "odd order thm" of Feit-Thompson: every simple group of odd order is of prime order)

<u>Lemma</u>: let $n \geq 5$. Then (1) All cycles of length 3 in A_n are conjugate, generate A_n.

(PS3)

(2) All elements of form $\tau_1\tau_2$, τ_i disjoint transpositions are conjugate, generate A_n.

(G is a gp, $x, y \in G$ are <u>conjugate</u> if $x = gyg^{-1}$ for some $g \in G$)

(G gp, $N \triangleleft G$, $y \in N$, $x \in G$, x, y conjugate then $x \in N$)

("normal" $\equiv$ $gNg^{-1} = N$)

<u>Conclusion</u>: If $N \triangleleft A_n$, N contains a 3-cycle or pdt of two transpositions then $N = A_n$. (If contain one, contain all)

<u>Pf of thm</u>: let $N \triangleleft A_n$, $N \neq \{id\}$. let $\sigma \in N$ be an element of minimal (but non-empty) support, wlog $\{1, \ldots, k\}$.

Divide into cases:

$k=1$ would make $\sigma = id$, $k=2$ would make $\sigma = (12) \notin A_n$

$k=3$ makes σ a 3-cycle, $N = A_n$. ✓

$k=4$ makes σ of the form $(12)(34)$ or $(1234) \notin A_n$ even cycles are odd permutation

so $N = A_n$ ✓

$k \geq 5$ and σ has a cycle of length ≥ 3 relabel so that this cycle begins $(123\ldots$

Consider $\rho = (345)\sigma(345)^{-1}\sigma^{-1}$. If $i > k, \geq 5$, $\sigma(i) = \sigma^{-1}(i) = i$

then $\rho(i) = i$, $\rho(2) = (345)\sigma(345)^{-1}\sigma^{-1}(2)$

$= 2$

$\rho(3) = 4$

so $\rho \neq id$, but $\operatorname{supp} \rho \not\supseteq \operatorname{supp} \sigma$.

Also $\rho \in N$: $(345)\sigma(345)^{-1} \in N$ since N is normal

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$k \geq 5$, σ is a pdt of disjoint transpositions. Then up to labelling,

$$\sigma = (12)(34)(56)(78)\ldots$$

Same $\rho = (345)\sigma(345)^{-1}\sigma^{-1}$ fixes $\operatorname{supp}(\sigma) \cup \{1,2\}$

but $\rho(7) = 8$.

Another contradiction.