

Math 322: Problem Set 4 (due 8/10/2015)

Practice Problems

- P1 Let G be a group with $|G| = 2$. Show that $G = \{e, g\}$ with $g \cdot g = e$ (hint: consider the multiplication table). Show that $G \simeq C_2$ (that is, find an isomorphism $C_2 \rightarrow G$).
- P2 Let G be a group. Give a bijection between $\{H < G \mid \#H = 2\}$ and $\{g \in G \mid g^2 = e, g \neq e\}$.
- P3 (Basics of groups and homomorphisms) Fix groups G, H, K and let $f \in \text{Hom}(G, H)$, .
- (a) Given also $g \in \text{Hom}(H, K)$, show that $g \circ f \in \text{Hom}(G, K)$.
 - (b) Suppose f is bijective. Then $f^{-1}: H \rightarrow G$ is a homomorphism.

Groups and Homomorphisms

1. Let G be a group, and let $(A, +)$ be an abelian group. For $f, g \in \text{Hom}(G, A)$ and $x \in G$ define $(f + g)(x) = f(x) + g(x)$ (on the right this is addition in A).
 - (a) Show that $f + g \in \text{Hom}(G, A)$.
 - (b) Show that $(\text{Hom}(G, A), +)$ is an abelian group.
 - (*c) Let G be a group, and let $\text{id}: G \rightarrow G$ be the identity homomorphism. Define $f: G \rightarrow G$ by $f(x) = (\text{id}(x))(\text{id}(x)) = x \cdot x = x^2$. Suppose that $f \in \text{Hom}(G, G)$. Show that G is commutative.
2. (External Direct products) Let G, H be groups.
 - (a) On the product set $G \times H$ define an operation by $(g, h) \cdot (g', h') = (gg', hh')$. Show that $(G \times H, \cdot)$ is a group.
 - DEF this is called the (external) *direct product* of G, H .
 - (b) Let $\tilde{G} = \{(g, e_H) \mid g \in G\}$ and $\tilde{H} = \{(e_G, h) \mid h \in H\}$. Show that \tilde{G}, \tilde{H} are subgroups of $G \times H$ and that $\tilde{G} \cap \tilde{H} = \{e_{G \times H}\}$.
 - SUPP Show that \tilde{G}, \tilde{H} are isomorphic to G, H respectively.
 - (c) Show that for any $x = (g, h) \in G \times H$ we have $x\tilde{G}x^{-1} = \tilde{G}$ and $x\tilde{H}x^{-1} = \tilde{H}$.
 - EXAMPLE The Chinese remainder theorem shows that $C_n \times C_m \simeq C_{nm}$ if $\text{gcd}(n, m) = 1$.
3. Products with more than two factors can be defined recursively, or as sets of k -tuples.
 - SUPP Find “natural” isomorphisms $G \times H \simeq H \times G$ and $(G \times H) \times K \simeq G \times (H \times K)$. We therefore write products without regard to the order of the factors.
 - DEF Write G^k for the k -fold product of groups isomorphic to G .
 - (a) Show that every non-identity element of C_2^k has order 2.
 - (b) Show that $C_3 \times C_3 \not\simeq C_9$.
4. The *Klein group* or the *four-group* is the group $V \simeq C_2 \times C_2$.
 - PRAC Check that $(\mathbb{Z}/12\mathbb{Z})^\times \simeq V$ and that $(\mathbb{Z}/8\mathbb{Z})^\times \simeq V$.
 - (a) Write a multiplication table for V , and show that V is not isomorphic to C_4 .
 - (b) Show that $V = H_1 \cup H_2 \cup H_3$ where $H_i \subset V$ are subgroups isomorphic to C_2 .
 - (c) Let G be a group of order 4. Show that G is isomorphic to either C_4 or to $C_2 \times C_2$.
5. Let G be a group, and let $H, K < G$ be subgroups and suppose that $H \cup K$ is a subgroup as well. Show that $H \subset K$ or $K \subset H$.

Bonus problems

6. Show that, for each $d|n$, $\mathbb{Z}/n\mathbb{Z}$ has a unique subgroup of order d (and that the subgroup is cyclic).
- 7**. Let G be a finite group of order n , and suppose that for each $d|n$ G has at most one subgroup of order d . Show that G is cyclic.

Supplementary Problems

- A. Let G be the *isometry group* of the Euclidean plane: $G = \{f: \mathbb{R}^n \rightarrow \mathbb{R}^n \mid \|f(\underline{x}) - f(\underline{y})\| = \|\underline{x} - \underline{y}\|\}$.
- (a) Show that every $f \in G$ is a bijection and that G is closed under composition and inverse.
- (b) For $\underline{a} \in \mathbb{R}^n$ set $t_{\underline{a}}(\underline{x}) = \underline{x} + \underline{a}$. Show that $t_{\underline{a}} \in G$, and that $\underline{a} \mapsto t_{\underline{a}}$ is an injective group homomorphism $(\mathbb{R}^n, +) \rightarrow G$.

DEF Call the image the subgroup of *translations* and denote it by T .

- (c) Let $K = \{g \in G \mid g(\underline{0}) = \underline{0}\}$. Show that $K < G$ is a subgroup (we usually denote it $O(n)$ and call it the *orthogonal group*).

DEF This is called the *orthogonal group* and consists of rotations and reflections.

FACT K acts on \mathbb{R}^n by linear maps.

- (d) Show $\forall g \in G \exists t \in T : g\underline{0} = t\underline{0}$, and hence that $t^{-1}g \in K$. Conclude that $G = TK$.
- (e) Show that every $g \in G$ has a *unique* representation in the form $g = tk$, $t \in T$, $k \in K$ (hint: what is $T \cap K$?)
- (f) Show that K *normalizes* T : if $k \in K$, $t \in T$ we have $ktk^{-1} \in T$ (hint: use the linearity of k).
- (g) Show that $T \triangleleft G$: for every $g \in G$ we have $gTg^{-1} = T$.

RMK We have shown that G is the *semidirect product* $G = K \rtimes T$.

- B. Let X be a set of size at least 2, and fix $e \in X$. Define $*$: $X \times X \rightarrow X$ by $x * y = y$.
- (a) Show that $*$ is an associative operation and that e is a left identity.
- (b) Show that every $x \in X$ has a right inverse: an element \bar{x} such that $x * \bar{x} = e$.
- (c) Show that $(X, *)$ is not a group.

- C. Let $\{G_i\}_{i \in I}$ be groups. On the cartesian product $\prod_i G_i$ define an operation by

$$(\underline{g} \cdot \underline{h})_i = g_i h_i$$

(that is, by co-ordinatewise multiplication).

- (a) Show that $(\prod_i G_i, \cdot)$ is a group.

DEF This is called the (external) *direct product* of the G_i .

- (b) Let $\pi_j: \prod_i G_i \rightarrow G_j$ be projection on the j th coordinate. Show that $\pi_j \in \text{Hom}(\prod_i G_i, G_j)$.
- (c) (Universal property) Let H be any group, and suppose given for each i a homomorphism $f_i \in \text{Hom}(H, G_i)$. Show that there is a unique homomorphism $\underline{f}: H \rightarrow \prod_i G_i$ such that for all i , $\pi_i \circ \underline{f} = f_i$.

- (**d) An *abstract direct product* of the groups G_i is a pair $(\mathbf{G}, \{q_i\}_{i \in I})$ where \mathbf{G} is a group, $q_i: \mathbf{G} \rightarrow G_i$ are homomorphisms, and the property of (c) holds. Suppose that \mathbf{G}, \mathbf{G}' are both abstract direct products of the same family $\{G_i\}_{i \in I}$. Show that \mathbf{G}, \mathbf{G}' are isomorphic (hint: the system $\{q_i\}$ and the universal property of \mathbf{G}' give a map $\phi: \mathbf{G} \rightarrow \mathbf{G}'$, and the same idea gives a map $\psi: \mathbf{G}' \rightarrow \mathbf{G}$. To see that the composition is the identity compare for example $q_i \circ \psi \circ \phi$, $q_i \circ \text{id}_{\mathbf{G}}$ and use the uniqueness of (c).

- D. Let V, W be two vector spaces over a field F . On the set of pairs $V \times W = \{(\underline{v}, \underline{w}) \mid \underline{v} \in V, \underline{w} \in W\}$ define $(\underline{v}_1, \underline{w}_1) + (\underline{v}_2, \underline{w}_2) = (\underline{v}_1 +_V \underline{v}_2, \underline{w}_1 +_W \underline{w}_2)$ and $a \cdot (\underline{v}_1, \underline{w}_1) = (a \cdot_V \underline{v}_1, a \cdot_W \underline{w}_1)$.
- (a) Show that this endows $V \times W$ with the structure of a vector space. This is called the *external direct sum* of V, W and denote it $V \oplus W$.
- (b) Generalize the construction to an infinite family of vector spaces as in problem C(a).
- (*c) State a universal property analogous to that of C(c), C(d) and prove the analogous results.
- E. (Supplement to P3) Let $S^1 \subset \mathbb{R}^2$ be the unit circle. Then $f: [0, 2\pi) \rightarrow S^1$ given by $f(\theta) = (\cos \theta, \sin \theta)$ is continuous, 1-1 and onto but its inverse is not continuous.