Math 101 - SOLUTIONS TO WORKSHEET 34 TAYLOR SERIES AND LIMITS

1. Derivatives

- (1) (Final 2014) Let ∑_{n=0}[∞] c_nxⁿ be the MacLaurin series for e^{3x}. Find c₅.
 Solution: Knowing that e^u = ∑_{n=0}[∞] uⁿ/n! we have e^{3x} = ∑_{n=0}[∞] 3ⁿ/n! xⁿ so c₅ = 3⁵/5!.
 (2) (Final 2013) Let f(x) = x² sin(x³). Find f¹¹(0).
 Solution: We know that sin u = u u³/3! + u⁵/5! · · · so

$$x^{2}\sin(x^{3}) = x^{2}\left(x^{3} - \frac{x^{9}}{3!} + \cdots\right) = x^{5} - \frac{x^{11}}{3!} + \cdots$$

It follows that $\frac{f^{(11)}(0)}{11!} = \frac{1}{3!}$ so $f^{(11)}(0) = \frac{11!}{3!}$.

- (3) Let $g(x) = \begin{cases} \frac{e^{-x^2} 1}{x} & x \neq 0\\ 0 & x = 0 \end{cases}$.
 - (a) Find $g^{(3)}(0)$.

Solution: Knowing that $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$, the series for e^{-x^2} begins $1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots$. We have

$$g(x) = \frac{-x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots}{x} = -x + \frac{1}{2}x^3 - \frac{1}{6}x^5 + \dots$$

By the formula for Taylor coefficients we have $\frac{g^{(3)}(0)}{3!} = \frac{1}{2}$ so $g^{(3)}(0) = 3$. (b) (2011 Final) Give the first three non-zero terms of the MacLaurin series for $\int g(x) dx$. **Solution:** Integrating term-by-term, the expansion of $\int g \, dx$ begins

$$C - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{36}x^6 + \cdots$$

2. LIMITS WITHOUT L'HÔPITAL'S RULE

(4) (Final 2012) Evaluate $\lim_{x\to 0} \frac{\sin(x) - x + x^3/6}{\sin(x^5)}$ **Solution:** We have $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots$ so $\sin x - x + \frac{1}{6}x^3 = \frac{1}{120}x^5 + \cdots$. From this we also get $\sin(x^5) = x^5 - \frac{1}{6}x^{15} + \cdots$. In summary we have:

$$\frac{\sin(x) - x + x^3/6}{\sin(x^5)} \approx \frac{\frac{1}{120}x^5}{x^5} = \frac{1}{120}$$

and hence

$$\lim_{x \to 0} \frac{\sin(x) - x + x^3/6}{\sin(x^5)} = \frac{1}{120}$$

(5) Evaluate $\lim_{x\to 0} \frac{x \sin x - \log(1+x^2)}{e^{-x^2/2} - \cos(x)}$ Solution: We have $\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots$ so $x \sin x \approx x^2 - \frac{1}{6}x^4$ to fourth order. Similarly, $\log(1+u) = u - \frac{u^2}{2} + \cdots$ so $\log(1+x^2) \approx x^2 - \frac{1}{2}x^4$ to fourth order. In the denominator we have

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$$\begin{aligned} e^u &= 1 + u + \frac{1}{2}u^2 + \dots \text{ so } e^{-\frac{1}{2}x^2} \approx 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 \text{ to fourth order. We have } \cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \text{ to fourth order. Correct to fourth order we therefore have} \\ \frac{x \sin x - \log(1 + x^2)}{e^{-x^2/2} - \cos(x)} \approx \frac{\left(x^2 - \frac{1}{6}x^4\right) - \left(x^2 - \frac{1}{2}x^4\right)}{\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4\right) - \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4\right)} = \frac{\left(\frac{1}{2} - \frac{1}{6}\right)x^4}{\left(\frac{1}{8} - \frac{1}{24}\right)x^4} = \frac{\frac{1}{2}\left(1 - \frac{1}{3}\right)}{\frac{1}{8}\left(1 - \frac{1}{3}\right)} = 4 \end{aligned}$$
 so that
$$\lim_{x \to 0} \frac{x \sin x - \log(1 + x^2)}{e^{-x^2/2} - \cos(x)} = 4. \end{aligned}$$