Math 101 – SOLUTIONS TO WORKSHEET 10
WORK, AVERAGE VALUE

(1) (Work)

(a) (Final, 2012) A tank in the shape of a hemispherical bowl of radius $R = 3$m is full of water. It is to be emptied through an outlet extending $H = 2$m above its top. Using the values $\rho = 1000\text{kg/m}^3$ for the density of water and $g = 9.8\text{m/s}^2$ for the acceleration due to gravity, find the work (in Joules) required to empty the tank completely. There is no need to simplify your answer but you must evaluate all integrals.

Solution: Slice the water into horizontal slabs perpendicular to the z-axis. Each slab has circular cross-section. The slab at height z above the bottom of the hemisphere is at distance $R - z$ below the centre of the sphere, so the radius of the slab satisfies

$$r^2 + (R - z)^2 = R^2$$

so

$$r^2 = 2Rz - z^2 \,.$$

The volume of the slab is therefore about $\pi r^2 \Delta z = \pi(2Rz - z^2)\Delta z$; the mass of the water in it is therefore

$$\Delta m \approx \pi\rho \left(2Rz - z^2\right) \Delta z \,.$$

This mass of water will be moved from the height z to the height $R + H$ against gravity, the required work is therefore

$$\begin{aligned} \Delta W &= \Delta m \cdot g \cdot (R + H - z) \\ &= \pi\rho g \left(2Rz - z^2\right)(R + H - z)\,\Delta z \,. \end{aligned}$$

The total work is therefore

$$\begin{aligned} \pi\rho g \int_{z=0}^{z=R} \left(2Rz - z^2\right)(R + H - z)\,\mathrm{d}z &= \pi\rho g \int_{z=0}^{z=R} \left(2Rz(R+H) - z^2(R+H) - 2Rz^2 + z^3\right)\mathrm{d}z \\ &= \pi\rho g \int_{z=0}^{z=R} \left(2R(R+H)z - (3R+H)z^2 + z^3\right)\mathrm{d}z \\ &= \pi\rho g \left[R(R+H)z^2 - \frac{3R+H}{3}z^3 + \frac{z^4}{4}\right]_{z=0}^{z=R} \\ &= \pi\rho g \left((R+H)R^3 - (R + \frac{H}{3})R^3 + \frac{R^4}{4}\right) \\ &= \pi\rho g R^3 \left(\frac{R}{4} + \frac{2}{3}H\right) \\ &= 9{,}800\pi \cdot 27 \left(\frac{3}{4} + \frac{4}{3}\right)\text{J} \,. \end{aligned}$$

Solution: Instead, let z measure depth below the surface of the water. Again we chop the water into horizontal slabs perpendicular to the z-axis. Then the slice at depth z has volume about $\pi r^2 \Delta z = \pi\left(R^2 - z^2\right)\Delta z$. This water will be moved distance $H + z$ against gravity, so

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the total work is

$$\begin{aligned}
\int_{z=0}^{R} \pi\rho g\left(R^2 - z^2\right)(H+z)\,\mathrm{d}z &= \pi\rho g \int_{z=0}^{R} \left(R^2 H + R^2 z - z^2 H - z^3\right)\mathrm{d}z \\
&= \pi\rho g\left[R^2 H z + \frac{R^2}{2}z^2 - \frac{H}{3}z^3 - \frac{z^4}{4}\right]_{z=0}^{z=R} \\
&= \pi\rho g\left[R^3 H + \frac{R^4}{2} - \frac{H}{3}R^3 - \frac{R^4}{4}\right] \\
&= \pi\rho g R^3\left(\frac{2}{3}H + \frac{R}{4}\right) \\
&= 9{,}800\pi\cdot 27\left(\frac{4}{3} + \frac{3}{4}\right)\mathrm{J}\,.
\end{aligned}$$

(b) (Final, 2013) A force of 10N (Newtons) is required to hold a spring stretched 5 cm beyond its natural length. How much work, in joules (J), is done in stretching the spring from its natural length to 50 cm beyond its natural length?

Solution: The spring constant is $k = \frac{10}{0.05}\frac{\mathrm{N}}{\mathrm{m}} = 200\frac{\mathrm{N}}{\mathrm{m}}$. Stretching the spring from extension x to extension $x + \mathrm{d}x$ would require force $F = kx$. The total work done would therefore be

$$W = \int_0^{0.5} F\,\mathrm{d}x = \int_0^{0.5} kx\,\mathrm{d}x = \left[\frac{1}{2}kx^2\right]_{x=0}^{x=0.5} = \frac{200}{8}\mathrm{Nm} = 15\mathrm{J}.$$