Math 101 – SOLUTIONS TO WORKSHEET Special
AVERAGE VALUE

1. Average Value

In this note I collect a few examples of computing the average value of a function, and some example problems using it.

Definition. Let f be defined and integrable on $[a,b]$. The *average value* of f on the interval is

$$\boxed{\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x}\,.$$

Remark 1. A Riemann sum for $\int_a^b f\,\mathrm{d}x$ is $\sum_{i=1}^n f(x_i^*)\Delta x = \sum_{i=1}^n f(x_i^*)\frac{b-a}{n}$; dividing by $b-a$ we see that a Riemann sum of the integral above is:

$$\frac{1}{n}\sum_{i=1}^n f(x_i^*)\,.$$

In other words, the average value of f on the interval is the limit of *averages of values of f at sample points*.

In straightforward problems you are given f, a, b and asked to compute the average. In more complicated problems a, b or f itself may depend on a parameter, and you need to have the confidence to compute the average in terms of the parameter, geting a formula instead of a numerical answer for the average value. You can then *solve* for the parameter using given information.

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2. Straight-up problems

In these problems, simply compute the average value of the given function on the given interval.

(1) $f(x) = e^{5x} + x\sqrt{x^2+1}$ on the interval $[-1, 2]$.

Solution: We plug in to the formula:

$$\bar{f} = \frac{1}{3}\int_{x=-1}^{x=2}\left(e^{5x} + x\sqrt{x^2+1}\right)\mathrm{d}x = \frac{1}{3}\int_{x=-1}^{x=2} e^{5x}\,\mathrm{d}x + \frac{1}{3}\int_{x=-1}^{x=2} x\sqrt{x^2+1}\,\mathrm{d}x\,.$$

For the first integral, $\frac{1}{5}e^{5x}$ is an obvious anti-derivative of e^{5x}. In the second one we use substitution, noting that for $u = x^2+1$ we have $\mathrm{d}u = 2x\,\mathrm{d}x$ so $x\,\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$ and:

$$\begin{aligned}\bar{f} &= \frac{1}{3}\left[\frac{1}{5}e^{5x}\right]_{x=-1}^{x=2} + \frac{1}{3}\cdot\frac{1}{2}\int_{u=2}^{u=5}\sqrt{u}\,\mathrm{d}u\\ &= \frac{1}{15}\left[e^{10} - e^{-5}\right] + \frac{1}{6}\left[\frac{2}{3}u^{3/2}\right]_{u=2}^{u=5}\\ &= \boxed{\frac{1}{15}\left(e^{10} - e^{-5}\right) + \frac{1}{9}\left(5^{3/2} - 2^{3/2}\right)}\,.\end{aligned}$$

(2) (Final, 2009) $f(\theta) = |\sin\theta - \cos\theta|$ on $\left[0, \frac{\pi}{2}\right]$.

Solution: On $\left[0, \frac{\pi}{4}\right]$ we have $\cos\theta \geq \sin\theta$, while the reverse is true on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$. We therefore have

$$\begin{aligned}\bar{f} &= \frac{1}{\pi/2}\int_0^{\pi/2} f(\theta)\,\mathrm{d}\theta = \frac{2}{\pi}\left(\int_0^{\pi/4}(\cos\theta - \sin\theta)\,\mathrm{d}\theta + \int_0^{\pi/4}(\sin\theta - \sin\theta)\,\mathrm{d}\theta\right)\\ &= \frac{2}{\pi}\left([\sin\theta + \cos\theta]_{\theta=0}^{\theta=\pi/4} + [-\cos\theta - \sin\theta]_{\theta=\pi/4}^{\theta=\pi/2}\right)\\ &= \frac{2}{\pi}\left(\left[\frac{2}{\sqrt{2}} - 1\right] + \left[-1 + \frac{2}{\sqrt{2}}\right]\right) = \frac{2}{\pi}\left(2\sqrt{2} - 2\right) = \boxed{\frac{4(\sqrt{2}-1)}{\pi}}\,.\end{aligned}$$

Solution: Our function is symmetric on relfection by $\frac{\pi}{4}$ since $f\left(\frac{\pi}{2} - \theta\right) = \left|\sin\left(\frac{\pi}{2} - \theta\right) - \cos\left(\frac{\pi}{2} - \theta\right)\right| = |\cos\theta - \sin\theta| = f(\theta)$. In particular, the average on $\left[0, \frac{\pi}{4}\right]$ and on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ are equal. It's therefore enough to compute the average on half the interval, which is

$$\begin{aligned}\frac{1}{\pi/4}\int_0^{\pi/4}(\cos\theta - \sin\theta)\,\mathrm{d}\theta &= \frac{4}{\pi}\left[\sin\theta + \cos\theta\right]_{\theta=0}^{\theta=\pi/4}\\ &= \frac{4}{\pi}\left[\frac{2}{\sqrt{2}} - 1\right] = \boxed{\frac{4(\sqrt{2}-1)}{\pi}}\,.\end{aligned}$$

(3) (Final, 2011) $f(x) = xe^x$ on $[0, 2]$.

Solution: We plug in to the formula and the integrate by parts, using $u = x$, $\mathrm{d}v = e^x\,\mathrm{d}x$ so that $v = e^x$, $\mathrm{d}u = \mathrm{d}x$.

$$\begin{aligned}\bar{f} &= \frac{1}{2}\int_0^2 xe^x\,\mathrm{d}x = \frac{1}{2}\left[xe^x\right]_{x=0}^{x=2} - \frac{1}{2}\int_0^2 e^x\,\mathrm{d}x\\ &= \frac{1}{2}2e^2 - \frac{1}{2}\left[e^2 - 1\right] = \boxed{\frac{e^2+1}{2}}\,.\end{aligned}$$

3. Problems involving a parameter

In the following problems, one piece of information (the function f or the interval) depends on a parameter. You need to compute the average value using the parameter, and then solve for the parameter.

(1) (Final, 2012) Let k be a positive constant. Find the average value of $f(x) = \sin(kx)$ on $[0, \pi/k]$.

Solution: The average value is

$$\begin{aligned} \bar{f} &= \frac{1}{\pi/k} \int_{x=0}^{x=\pi/k} \sin(kx)\,\mathrm{d}x \\ &= \frac{1}{\pi} \int_{x=0}^{x=\pi/k} \sin(kx) k\,\mathrm{d}x \\ &\overset{u=kx}{=} \frac{1}{\pi} \int_{u=0}^{u=\pi} \sin(u)\,\mathrm{d}u = \frac{1}{\pi} \left[-\cos u\right]_{u=0}^{u=\pi} \\ &= \frac{1}{\pi} \left[-(-1) - (-(1))\right] = \boxed{\frac{2}{\pi}}. \end{aligned}$$

(2) Let $f(x) = x\sqrt{x^2 + r^2}$. For what value of $r > 0$ is the average value of f on $[0, 3]$ equal to $\frac{1}{9}$?

Solution: Substituting $u = x^2 + r^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$, the average value is

$$\begin{aligned} \bar{f} &= \frac{1}{3} \int_{x=0}^{x=3} x\sqrt{x^2 + r^2}\,\mathrm{d}x \\ &= \frac{1}{3} \cdot \frac{1}{2} \int_{x=0}^{x=3} \sqrt{u}\,\mathrm{d}u = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{2}{3} \left[u^{3/2}\right]_{x=0}^{x=2} \\ &= \frac{1}{9} \left[\sqrt{9 + r^2} - \sqrt{r^2}\right]. \end{aligned}$$

We are given the averaeg value is $\frac{1}{9}$, so we have

$$\frac{1}{9}\left(\sqrt{9 + r^2} - r\right) = \frac{1}{9},$$

hence

$$\begin{aligned} \sqrt{9 + r^2} &= r + 1 \\ 9 + r^2 &= (r + 1)^2 = r^2 + 2r + 1 \\ 9 &= 2r + 1 \\ 4 &= r\,. \end{aligned}$$

(3) (Final, 2010) Find a number $b > 0$ such that the function $f(x) = x - 1$ has average value 0 on the interval $[0, b]$.

Solution: The average value on the interval is $\frac{1}{b} \int_0^b f(x)\,\mathrm{d}x$. We don't know the value of b yet, but that shouldn't stop us from evaluating this expression. We get:

$$\begin{aligned} \frac{1}{b} \int_0^b f(x)\,\mathrm{d}x &= \frac{1}{b} \int_0^b (x - 1)\,\mathrm{d}x = \frac{1}{b} \left[\frac{x^2}{2} - x\right]_{x=0}^{x=b} \\ &= \frac{1}{b} \left[\frac{1}{2}b^2 - b\right] = \frac{1}{2}b - 1 \end{aligned}$$

We see that the average value is zero when $\frac{1}{2}b - 1 = 0$, that is when $\boxed{b = 2}$.