MATH 100 – WORKSHEET 21
OPTIMIZATION

Problem-solving steps: (0) read carefully (1) Draw picture, fix coordinate system; (2) parametrize; (3) Enforce relations; (4) Calculus; (5) Endgame.

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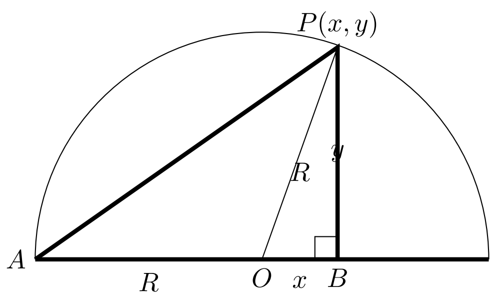
(1) (Final 2012) The right-angled triangle ΔABP (AP is the hypotenuse) has the vertex $A = (-1, 0)$, the vertex P lie on the semicircle $y = \sqrt{1-x^2}$ and the vertex B on the x-axis. What is the largest possible area of this triangle?

Solution

(1) Picture:



(2) Put the coordinate system where the centre of the circle is at $(0, 0)$ and the diameter is on the x-axis. Let B be at $(x, 0)$, P at (x, y).

(3) Then (P on the circle) $y = \sqrt{1-x^2}$ and the area of the circle is $A = \frac{1}{2}(\text{base}) \times (\text{height}) = \frac{1}{2}(1+x)\sqrt{1-x^2}$ since the base of the triangle has length $1 + x$.

(4) The function $A(x)$ is continuous on $[-1, 1]$ so we can find its minimum by differentiation. By the product rule and chain rule,

$$\begin{aligned} A'(x) &= \frac{1}{2}\sqrt{1-x^2} + \frac{1}{2}(1+x)\frac{-2x}{2\sqrt{1-x^2}} \\ &= \frac{\left(\sqrt{1-x^2}\right)^2}{2\sqrt{1-x^2}} - \frac{x(1+x)}{2\sqrt{1-x^2}} = \frac{1-x^2-x-x^2}{2\sqrt{1-x^2}} \\ &= \frac{1-x-2x^2}{2\sqrt{1-x^2}}. \end{aligned}$$

This is defined on $(-1, 1)$ and at critical points we have $2x^2+x-1 = 0$, that is there $x = \frac{-1\pm\sqrt{1+8}}{4} = \frac{-1\pm 3}{4} = -1, \frac{1}{2}$. We therefore have singularities at the endpoints ± 1 and a critical point at $x = \frac{1}{2}$. The area vanishes at the endpoints (the triangle becomes degenerate) and

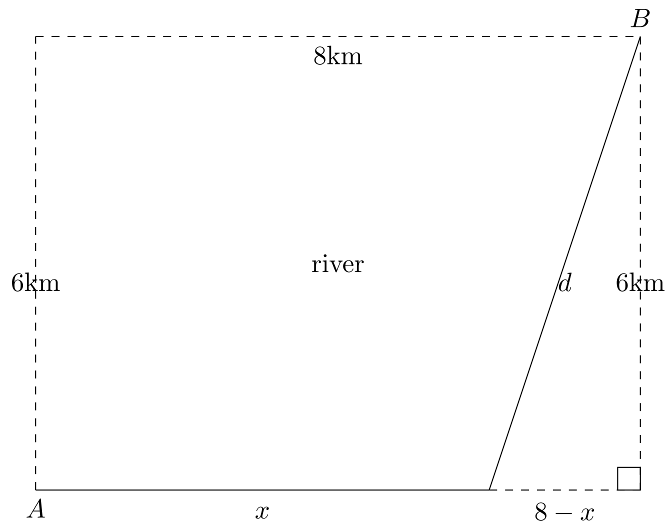
$$A\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \frac{3}{2} \cdot \sqrt{1 - \frac{1}{2^2}} = \frac{3\sqrt{3}}{8}.$$

It follows that the largest possible area is $\frac{3\sqrt{3}}{8}$.

(2) (Final 2010) A river running east-west is 6km wide. City A is located on the shore of the river; city B is located 8km to the east on the opposite bank. It costs \$40/km to build a bridge across the river, \$20/km to build a road along it. What is the cheapest way to construct a path between the cities?

Solution

(1) Picture:



(2) Build a road of length x from A along the bank, then build a bridge of length d toward B.

(3) By Pythagoras, $d = \sqrt{6^2 + (8-x)^2}$ so the total cost is

$C(x) = 20x + 40\sqrt{6^2 + (8-x)^2} = 20x + 40\sqrt{6^2 + (x-8)^2}$.

(4) The function $A(x)$ is defined everywhere ($6^2 + (8-x)^2 \geq 6^2 > 0$) and continuous there. We have

$$A'(x) = 20 + 40\frac{2(x-8)}{2\sqrt{6^2 + (x-8)^2}}.$$

This exists everywhere (the denominator is everywhere positive by the same calculation). It's enough to consider $0 \leq x \leq 8$ (no point in starting the bridge west of A or east of B). Looking for critical points there we solve $A'(x) = 0$ that is:

$$\begin{aligned}
20 + 40\frac{x-8}{\sqrt{36 + (x-8)^2}} &= 0 \\
20 &= 40\frac{8-x}{\sqrt{36 + (8-x)^2}} \\
\sqrt{36 + (8-x)^2} &= 2(8-x) \\
36 + (8-x)^2 &= 4(8-x)^2 \\
36 &= 3(8-x)^2 \\
(8-x) &= \sqrt{\frac{36}{3}} = \sqrt{12} = 2\sqrt{3}
\end{aligned}$$

(only the positive root since $0 \le x \le 8$ forces $8 - x \ge 0$) so

$$x = 8 - 2\sqrt{3}\,.$$

We then have $A(0) = 40\sqrt{6^2 + 8^2} = 40\sqrt{100} = 400$, $A(8) = 20 \cdot 8 + 40\sqrt{6^2} = 160 + 240 = 400$ and

$$\begin{aligned} A(8-2\sqrt{3}) &= 20\left(8 - 2\sqrt{3}\right) + 40\sqrt{6^2 + (2\sqrt{3})^2} = 160 - 40\sqrt{3} + 40\sqrt{36+12} \\ &= 160 - 40\sqrt{3} + 40\sqrt{48} = 160 - 40\sqrt{3} + 40\sqrt{16 \cdot 3} \\ &= 160 - 40\sqrt{3} + 40 \cdot 4\sqrt{3} = 160 + 120\sqrt{3}\,. \end{aligned}$$

Now $\sqrt{3} < \sqrt{4} = 2$ so $A(8 - 2\sqrt{3}) = 160 + 120\sqrt{3} < 160 + 120 \cdot 2 = 400 = A(0) = A(8)$ and we conclude that $A(8 - 2\sqrt{3})$ is the minimum.

(5) The cheapest way to construct a bridge is construct a road of length $\left(8 - 2\sqrt{3}\right)$ km along the bank from A toward B, and then bridge from the end of the road to B.