MATH 100 – SOLUTION TO WORKSHEET 12
EXPONENTIAL GROWTH AND DECAY

1. Exponentials

(1) A pair of invasive Opossums arrives in BC in 1930. Unchecked, Opossums can triple their population annually. Assume that this in fact happens.

(a) At what time will there be 1000 Opossums in BC? 10,000 Opossums?
Solution: (parametrize) Suppose that t years after 1930 there are $N(t)$ possums present. Each year the population triples, and $N(0) = 2$ so

$$N(t) = 2 \cdot 3^t \,.$$

(solve) $N(t) = 1000$ occurs when $2 \cdot 3^t = 1000$ so $3^t = 500$ so $t \log 3 = \log 500$, $t = \frac{\log 500}{\log 3}$.
(endgame) We conclude that there were 1000 possums $\boxed{\dfrac{\log 500}{\log 3}}$ years after 1930. For the same reasons there were 10,000 possums $\boxed{\dfrac{\log 5000}{\log 3}}$ years after 1930.

(b) Write a differential equation expressing the growth of the Opossum population with time.
Solution: $\frac{\mathrm{d}N}{\mathrm{d}t} = 2 \cdot 3^t \log 3 = (\log 3) \cdot N$.

(2) A radioactive sample decays according to the law

$$\frac{\mathrm{d}m}{\mathrm{d}t} = km \,.$$

(a) Suppose that one-quarter of the sample remains after 10 hours. What is the half-life?
Solution 1: The solution to the equation is exponential decay; $m(t) = Ce^{kt}$ for some C, k. We are given that $m(10) = \frac{1}{4}m(0)$. Thus

$$Ce^{10k} = \frac{1}{4}C$$

and, cancelling the C and taking log,

$$10k = \log\frac{1}{4} = -\log 4 \,.$$

Thus

$$k = -\frac{\log 4}{10}$$

(*sanity check*: k is negative, showing exponential decay). Now we need to find $t_{1/2}$ where $m(t_{1/2}) = \frac{1}{2}m(0)$, that is:

$$Ce^{kt_{1/2}} = \frac{1}{2}C \,.$$

This happens when $kt_{1/2} = \log\frac{1}{2} = -\log 2$ so

$$t_{1/2} = -\frac{\log 2}{\log 4/10} = \frac{10\log 2}{2\log 2} = 5 \text{ hours.}$$

Solution 2: After 10 hours we see two halvings of the sample (we now have a quarter) so one halving takes 5 hours and $t_{1/2} = 5$ hours.

Date: 20/10/2015, Worksheet by Lior Silberman.

(b) A 100-gram sample is left unattended for three days. How much of it remains?
Solution: the mass of the sample is $m(t) = Ce^{kt}$. We know $C = m(0) = 100$ and $k = -\frac{\log 4}{10}$ so after 72 hours we have:

$$\begin{aligned} m(3 \text{ days}) &= 100e^{-\frac{\log 4}{10}72}\text{gram} \\ &= 100e^{-7.2\log 4}\text{gram} \end{aligned}$$

2. Newton's Law of Cooling

Example (Final Exam, 2010)**.** When an apple is taken from a refrigerator, its temperature is $3°C$. After 30 minutes in a $19°C$ room its temperature is $11°C$.

(1) Write the *differential equation* satisfied by the temperature $T(t)$ of the apple.
(2) Find the temperature of the apple 90 minutes after it is taken from the refrigerator, expressed as an integer number of degrees Celsius.
(3) Determine the time when the temperature of the apple is $16°C$.

Solution

(1) Let $y(t) = T_{\text{apple}} - T_{\text{env}}$ denote the *temperature difference* between the apple and the room. By NLC, $y(t)$ decays exponentially so $y(t) = Ce^{kt}$ for some constants C, k to be determined. In degrees Celsius and with time in minutes, $y(0) = 3 - 19 = -16$ and $y(30) = 11 - 19 = -8$. Thus $\boxed{C = y(0) = -16}$ and

$$-8 = y(30) = -16e^{30k}$$

gives

$$e^{30k} = \frac{-8}{-16} = \frac{1}{2}$$

so

$$30k = \log\frac{1}{2} = -\log 2$$

and

$$\boxed{k = -\frac{\log 2}{30}.}$$

The differential equation of Newton's law of cooling is $T' = k\,(T - T_{\text{env}})$ which in our case is

$$T' = -\frac{\log 2}{30}\left(T - 19°C\right).$$

Remark. A common error is to forget the conversion to temperature differences, claiming that $T(t) = Ce^{kt}$ with $C = 3$ and $3e^{30k} = 11$. One way to notice you've made this error is to note that you get $30k = \log\frac{11}{3}$ so that $k = \frac{1}{30}\log\frac{11}{3}$ so that $\lim_{t\to\infty} T(t) = \infty$ which makes no sense.

(2) Plugging in we get

$$\begin{aligned} y(90) &= -16e^{-\frac{\log 2}{30}90\cdot} = -16e^{-3\log 2} \\ &= -16\left(e^{\log 2}\right)^{-3} = -16(2)^{-3} \\ &= -16/8 = -2°C\,. \end{aligned}$$

So after 90 minutes the apple is 2 degrees below room temperature, or

$$\boxed{T(90) = 17°C\,.}$$

Remark. Common erros include (1) failing to simplify (keeping $-16e^{-3\log 2}$) and (2) giving $y(90)$ instead of $T(90)$.

(3) We need to find t such that $Ce^{kt} = -3°C$, that is

$$\begin{aligned}
-16e^{-\frac{\log 2}{30}t} &= -3 \\
e^{-\frac{\log 2}{30}t} &= \frac{3}{16} \\
-\frac{\log 2}{30}t &= \log\frac{3}{16} = -\log\frac{16}{3}
\end{aligned}$$

and hence

$$\boxed{t = \frac{\log 16 - \log 3}{\log 2} 30 \text{ minutes},}$$