Math 322: Problem Set 6 (due 23/10/2014)

Practice problems

P1. Let G be a group and let X be a set of size at least 2. Fix $x_0 \in X$ and for $g \in G, x \in X$ set $g \cdot x = x_0$.

(a) Show that this operation satisfies $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G, x \in X$.

(b) This is not a group action. Why?

P2. Label the elements of the four-group V by $1, 2, 3, 4$ in some fashion, and explicitely give the permutation corresponding to each element by the regular action.

P3. Repeat with S_3 acting on itself by conjugation (you will now have six permutations in S_6).

P4. Let G act on X. Say that $A \subset X$ is *G-invariant* if for every $g \in G$, $a \in A$ we have $g \cdot a \in A$.

(a) Show that A is G-invariant iff $g \cdot A = A$ ($g \cdot A$ in the sense of problem 4(a)).

(b) Suppose A is G-invariant. Show that the restriction of the action to A (formally, the binary operation $\cdot \restriction_{G \times A}$) is an action of G on A.

Simplicity of A_n

1. Let $V = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$. Show that $V \lhd S_4$, so that S_4 is not simple.

2. (The normal subgroups of S_n) Let $N \lhd S_n$ with $n \geq 5$.

(a) Let G be a group and let $H \lhd G$ be a normal subgroup isomorphic to C_2. Show that $H < Z(G)$ (hint: let $H = \{1, h\}$, let $g \in G$, and consider the element ghg^{-1}).

(b) Suppose that $N \cap A_n \neq \{\mathrm{id}\}$. Show that $N \supset A_n$ and conclude that $N = A_n$ or $N = S_n$ (hint: what is the index of N?)

(c) Suppose that $N \cap A_n = \{\mathrm{id}\}$. Show that N is isomorphic to a subgroup of C_2 (hint: restrict $\mathrm{sgn}\colon S_n \to C_2$ to N).

(d) Show that if $n \geq 3$ then $Z(S_n) = \{\mathrm{id}\}$, and conclude that in case (c) we must have $N = \mathrm{id}$.

3. Let X be an infinite set.

(a) Show that $S_X^{\mathrm{fin}} = \{\sigma \in S_X \mid \mathrm{supp}(\sigma) \text{ is finite}\}$ is a subgroup of S_X.

PRAC For finite $F \subset X$ there is a natural inclusion $S_F \hookrightarrow S_X$, which is a group homomorphism, an isomorphism onto its image. Let $\mathrm{sgn}_F\colon S_F \to \{\pm 1\}$ be the sign character.

DEF For $\sigma \in S_X^{\mathrm{fin}}$ define $\mathrm{sgn}(\sigma) = \mathrm{sgn}_F(\sigma)$ for any finite F such that $\sigma \in S_F$.

(c) Show that $\mathrm{sgn}(\sigma)$ is well-defined (independent of F) and a group hom $S_X^{\mathrm{fin}} \to \{\pm 1\}$.

(d) The *infinite alternating group* A_X is kernel of this homomorphism. Show that A_X is simple.

Group actions

4. Let the group G act on the set X.

(a) For $g \in G$ and $A \in P(X)$ set $g \cdot A = \{g \cdot a \mid a \in A\} = \{x \in X \mid \exists a \in A : x = g \cdot a\}$. Show that this defines an action of G on $P(X)$.

(b) In PS2 we endowed $P(X)$ with a group structure. Show that the action of (a) is by *automorphism*: that the map $A \mapsto g \cdot A$ is a group homomorphism $(P(X), \Delta) \to (P(X), \Delta)$.

(c) Let Y be another set. For $f\colon X \to Y$ set $(g \cdot f)(x) = f(g^{-1} \cdot x)$. Show that this defines an action of G on Y^X, the set of functions from X to Y.

(*d) Suppose that $Y = \mathbb{R}$ (or any field), so that $\mathbb{R}^X$ has the structure of a vector space over $\mathbb{R}$. Show that the action of (c) is by *linear maps*.

5. (Some stabilizers) The action of S_X on X induces an action on $P(X)$ as in problem 3(a). Suppose that X is finite, $\#X = n$.
   (a) Show that the orbits of S_X on $P(X)$ are exactly the sets $\binom{X}{k} = \{A \subset X \mid \#A = k\}$.
   SUPP When X is infinite, $\binom{X}{\kappa}$ are orbits if $\kappa < |X|$, but there are multiple orbits on $\binom{X}{|X|}$, parametrized by the cardinality of the complement.
   (b) Let $A \subset X$. Show that $\mathrm{Stab}_{S_X}(A) \simeq S_A \times S_{X-A}$.
   (c) Use (a),(b) to show that $\#\binom{X}{k} = \frac{n!}{k!(n-k)!}$.

Conjugation

6. Let G be a finite group, H a proper subgroup. Show that the conjugates of H do not cover G (that is, there is some $g \in G$ which is not conjugate to an element of H).
   RMK There exists an infinite group in which all non-identity elements are conjugate.

Supplement: Cyclic Groups

A. Let $H \subset \mathbb{Z}/n\mathbb{Z}$ be a subgroup other than $\{\overline{0}\}$.
   (a) Let a be the smallest positive integer such that $\bar{a} \in H$. Show that $H = \{\overline{ma} \mid m \in \mathbb{Z}\}$.
   (b) Let $d = \gcd(a,n)$. Show that $\bar{d} \in H$ and conclude that a divides n.
   (c) Conversely, show that if $n = ab$ then $\left\{\bar{0}, \bar{a}, \overline{2a}, \cdots, \overline{(b-1)a}\right\}$ is a subgroup of $\mathbb{Z}/n\mathbb{Z}$.
   (d) Conclude that $\mathbb{Z}/n\mathbb{Z}$ has exactly one subgroup of order d for each $d|n$.

B. In this problem we show the converse to the previous one: if G is a finite group having at most one subgroup of every order then G is isomorphic to C_n.
   (a) For the rest of the problem, let G be the smallest group satisfying the hypothesis which is not cyclic. Show that every proper subgroup of G is cyclic.
   (b) Let d be a proper divisor of $n = |G|$. Show that G has $\phi(d)$ elements of order exactly d.
   (c) Show that G has elements of order n, and is therefore cyclic.