Math 539: Problem Set 2 (due 10/3/2014)

Dirichlet Characters

0. List all Dirichlet characters mod 15 and mod 16. Determine which are primitive.

1. Fix $q > 1$.
   (a) Let χ be a non-principal Dirichlet character mod q. Show that $\sum_p \frac{\chi(p)}{p}$ converges.
   (b) Let $(a,q) = 1$. Show that $\sum_{p\equiv a(q), p\leq x} \frac{1}{p} = \frac{1}{\varphi(q)} \log\log x + O(1)$
   (*c) Improve the error term to $C + O\left(\frac{1}{\log x}\right)$.

Dirichlet Series

2. (Convergence of Dirichlet series) Let $D(s) = \sum_{n\geq 1} a_n n^{-s}$ be a formal Dirichlet series. We will study the convergence of this series as s varies in $\mathbb{C}$.
   (a) Suppose that $D(s)$ converges absolutely at some $s_0 = \sigma_0 + it$. Show that $D(s)$ converges absolutely in the closed half-plane $\Re(s) = \sigma \geq \sigma_0$, uniformly in every half-plane of the form $\sigma \geq \sigma_1 > \sigma_0$.
   (b) Conclude that there is an *abcissa of absolute convergence* $\sigma_{\mathrm{ac}} \in [-\infty, +\infty]$ such that one of the following holds: (1) ($\sigma_{\mathrm{ac}} = \infty$) $D(s)$ does not converge absolutely for any $s \in \mathbb{C}$; (2) ($\sigma_{\mathrm{ac}} \in (-\infty, +\infty)$)$D(s)$ converges absolutely exactly in the half-plane $\sigma > \sigma_{\mathrm{ac}}$ or $\sigma \geq \sigma_{\mathrm{ac}}$; (3) ($\sigma_{\mathrm{ac}} = -\infty$) $D(s)$ converges absolutely in $\mathbb{C}$. In cases (2),(3) the convergence is uniform in any half-plane whose closure is a proper subset of the domain of convergence.
   (c) Suppose that $D(s)$ converges at some s_0. Show that $D(s)$ converges in the open half-plane $\sigma > \sigma_0$, locally uniformly in every half-plane of the form $\sigma \geq \sigma_1 > \sigma_0$, and that $D(s)$ converges absolutely in the half-plane $\sigma > \sigma_0 + 1$.
   (d) Conclude that there is an *absicssa of convergence* $\sigma_{\mathrm{c}} \in [-\infty, \infty]$ such that on of the following holds: (1) ($\sigma_{\mathrm{c}} = \infty$) $D(s)$ does not converge for any $s \in \mathbb{C}$; (2) ($\sigma_{\mathrm{c}} \in (-\infty, +\infty)$)$D(s)$ converges in the open half-plane $\sigma > \sigma_{\mathrm{c}}$ and diverges in the open half-plane $\sigma < \sigma_{\mathrm{c}}$; the convergence is locally uniform in any half-plane $\sigma \geq \sigma_1 > \sigma_{\mathrm{c}}$ (3) ($\sigma_{\mathrm{ac}} = -\infty$) $D(s)$ converges absolutely in $\mathbb{C}$. In cases (2) the convergence is uniform in any half-plane. Furthermore, σ_{c} and σ_{ac} are either both $-\infty$, both $+\infty$, or both finite, and in the latter case $\sigma_{\mathrm{c}} \leq \sigma_{\mathrm{ac}} \leq \sigma_{\mathrm{c}} + 1$.

3. Let $D(s)$ have abcissa of absolute convergence σ_{ac}.
   (a) Suppose $\sigma_{\mathrm{ac}} \geq 0$. Show that $\sum_{n\leq x} |a_n| \ll_\varepsilon x^{\sigma_{\mathrm{ac}}+\varepsilon}$.
   (b) Suppose $\sigma_{\mathrm{ac}} < 1$. Show that $\sum_{n>x} |a_n| n^{-1} \ll_\varepsilon x^{\sigma_{\mathrm{ac}}+\varepsilon}$

4. (Convergence of sums and products) Let $D_1(s) = \sum_{n\geq 1} a_n n^{-s}$ and $D_2(s) = \sum_{n\geq 1} b_n n^{-s}$, and let $(D_1 + D_2)(s) = \sum_{n\geq 1} (a_n + b_n) n^{-s}$, $(D_1 \cdot D_2)(s) = \sum_{n\geq 1} c_n n^{-s}$ where $c = a * b$ is the Dirichlet convolution.
   (a) Show that the domain of absolute convergence of $D_1 + D_2$ and $D_1 D_2$ is at least the intersection of the domains of absolute convergence of D_1, D_2.
   HARD (Mertens) Suppose that D_1, D_2 have abcissa of convergence σ_{c}. Show that $D_1 D_2$ has abcissa of convergence at most $\sigma_c + \frac{1}{2}$.

5. (Uniqueness of Dirichlet series) Suppose that $D(s)=\sum_{n\geq1}a_nn^{-s}$ converges somewhere
   (a) Suppose that $a_n=0$ if $n<N$ and $a_N\neq0$. Show that $\lim_{\Re(s)\to\infty}N^sD(s)=a_N$.
   (b) Suppose that $D_2(s)=\sum_{n\geq1}b_nn^{-s}$ also converges somewhere, and that $D(s_k)=D_2(s_k)$ for $\{s_k\}$ in the common domain of convergence such that $\lim_{k\to\infty}\Re(s_k)=\infty$. Show that $a_n=b_n$ for all n.

6. (Landau's Theorem; proof due to K. Kedlaya) Let $D(s)=\sum_{n\geq1}a_nn^{-s}$ have non-negative coefficients.
   (a) Show that $\sigma_{\mathrm{c}}=\sigma_{\mathrm{ac}}$ for this series.
   (b) Suppose that $D(s)$ extends to a holomorphic function in a small ball $|s-\sigma_{\mathrm{c}}|<\varepsilon$. Show that if $s<\sigma_c<\sigma$ and s,σ are close enough to σ_{c} then s is in the domain of convergence of the Taylor expansion of D at σ.
   (c) Using that $D^{(k)}(\sigma)=\sum_{n=1}^{\infty}a_n(-\log n)^kn^{-\sigma}$, write $D(s)$ as the sum of a two-variable series with positive terms.
   (d) Changing the order of summation, show that $D(s)$ converges at s, a contradiction to the definition of σ_{c}.
   (e) Obtain *Landau's Theorem*: if $D(s)$ has positive coefficients, has abcissa of convergence σ_{c}, and agrees with a holomorphic function in some punctured neighbourhood of σ_{c} then the singularity at $s=\sigma_{\mathrm{c}}$ is not removable.

Fourier Analysis

7. (Basics of Fourier series)
   (a) Let $D_N(x)=\sum_{|k|\leq N}e(kx)$ be the Dirichlet kernel. Show that $\int_0^1|D_N(x)|\,\mathrm{d}x\gg\log N$.
   (b) Let $F_N(x)=\sum_{|k|<N}\left(1-\frac{|k|}{N}\right)e(kx)$ be the Fejér kernel. Show that for $\delta\leq|x|\leq\frac{1}{2}$, we have $|F_N(x)|\leq\frac{1}{N\sin^2(\pi\delta)}$ so that for $f\in L^1(\mathbb{R}/\mathbb{Z})$,
$$\lim_{N\to\infty}\int_{\delta\leq|x|\leq\frac{1}{2}}|f(x)|\,|F_N(x)|\,\mathrm{d}x=0\,.$$
   (c) In class we showed that "smoothness implies decay": if $f\in C^r(\mathbb{R}/\mathbb{Z})$ then for $k\neq0$, $\left|\hat{f}(k)\right|\ll_r\|f\|_{C^r}|k|^{-r}$. Show the following partial converse: if $\left|\hat{f}(k)\right|=O(k^{-r-\varepsilon})$ then $\sum_{k\in\mathbb{Z}}\hat{f}(k)e(kx)\in C^{r-1}(\mathbb{R}/\mathbb{Z})$.

8. (The Basel problem) Let $f(x)$ be the $\mathbb{Z}$-periodic function on $\mathbb{R}$ such that $f(x)=x^2$for $|x|\leq\frac{1}{2}$.
   (a) Find $\hat{f}(k)$ for $k\in\mathbb{Z}$.
   (b) Show that $\zeta(2)=\frac{\pi^2}{6}$.
   (c) Apply *Parseval's identity* $\|f\|_{L^2(\mathbb{R}/\mathbb{Z})}=\left\|\hat{f}\right\|_{L^2(\mathbb{Z})}$ to evaluate $\zeta(4)$.

9. Let $\varphi\in\mathcal{S}(\mathbb{R})$.
   (a) Let $c\in L^2(\mathbb{Z}/q\mathbb{Z})$. Show that $\sum_{n\in\mathbb{Z}}c(n)\varphi(n)=\sum_{k\in\mathbb{Z}}\hat{c}(-k)\hat{\varphi}(k/q)$.
   (b) Let χbe a primitive Dirichlet character mod q. Show that
$$\sum_{n\in\mathbb{Z}}\chi(n)\varphi(n)=\frac{G(\chi)\chi(-1)}{q}\sum_{k\in\mathbb{Z}}\bar{\chi}(k)\hat{\varphi}\left(\frac{k}{q}\right).$$

Application: Weyl differencing and equidistribution on the circle

10. (Equidistribution) Let X be a compact space, μ a fixed probability measure on X (thought of as the "uniform" measure). We say that a sequence of probability measures $\{\mu_n\}_{n=1}^{\infty}$ is *equidistributed* if it converges to μ in the weak-* sense, that is if for every $f \in C(X)$, $\lim_{n\to\infty} \mu_n(f) = \mu(f)$ (equivalently, if for every open set $U \subset X$, $\mu_n(U) \to \mu(U)$).
   (a) Show that it is enough to check convergence on a set $B \subset C(X)$ such that $\mathrm{Span}_{\mathbb{C}}(B)$ is dense in $C(X)$.
   (b) (Weyl criterion) We will concentrate on the case $X = \mathbb{R}/\mathbb{Z}$, $\mu =$ Lebesgue. Show that in that case it is enough to check whether $\int_0^1 e(kx)\,\mathrm{d}\mu_n(x) \xrightarrow[n\to\infty]{} 0$ for each non-zero $k \in \mathbb{Z}$. (Hint: Stone–Weierstrass)
   DEF We say that a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is equidistributed (w.r.t. μ) if the sequence $\left\{\frac{1}{n}\sum_{k=1}^{n}\delta_{x_k}\right\}_{k=1}^{\infty}$ is equidistributed, that is if for every open set U the proportion of $1 \le k \le n$ such that $x_k \in U$ converges to $\mu(U)$, the proportion of the mass of X carried by μ.
   (c) Let α be irrational. Show directly that the sequence $n\alpha$ is dense in $[0,1]$.
   (d) Let α be irrational. Show that the sequence of fractional parts $\{n\alpha \mod 1\}_{n=1}^{\infty}$ is equidistributed in $[0,1]$.
   (e) Returning to the setting of parts (a),(b). suppose that $\mathrm{supp}(\mu) = X$. Show that every equidistributed sequence is dense.