Math 412: Problem set 10, due 7/4/2014

Differenetial Equations

1. We will analyze the differential equation $u'' = -u$ with initial data $u(0) = u_0$, $u'(0) = u_1$.
   (a) Let $\underline{v}(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}$. Show that u is a solution to the equation iff $\underline{v}$ solves
   $$\underline{v}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \underline{v}(t)\,.$$
   (b) Let $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Find formulas for W^n and express $\exp(Wt) = \sum_{k=0}^{\infty} \frac{W^k t^k}{k!}$ as a matrix whose entries are standard power series.
   (c) Show that $u(t) = u_0 \cos(t) + u_1 \sin(t)$.
   (d) Find a matrix S such that $W = S \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} S^{-1}$. Evaluate $\exp(Wt)$ again, this time using $\exp(Wt) = S \left(\exp \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} \right) S^{-1}$.

2. Consider the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\underline{v} = B\underline{v}$where B is at in PS7 problem 1.
   (a) Find matrices S, D so that D is in Jordan form, and such that $B = SDS^{-1}$.
   (b) Find $\exp(tD)$ directly (as in 1(b)).
   (c) Find the solution such that $\underline{v}(0) = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}^{\mathrm{t}}$.

Power series

3. Products of absolutely convergent series.
   (a) Let V be a normed space, and let $T, S \in \mathrm{End}_{\mathrm{b}}(V)$ commute. Show that $\exp(T+S) = \exp(T)\exp(S)$.
   (b) Show that, for appropriate values of t, $\exp(A)\exp(B) \neq \exp(A+B)$ where $A = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ -t & 0 \end{pmatrix}$.

Companion matrices

PRAC Find the Jordan canonical form of $\begin{pmatrix} & 1 & \\ & & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

4. Let $C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}$ be the companion matrix associated with the polynomial $p(x) = x^n - \sum_{k=0}^{n-1} a_k x^k$.
   (a) Show that $p(x)$ is, indeed, the characteristic polynomial of C.
   – For parts (b),(c) fix a non-zero root λ of $p(x)$.
   (b) Find (with proof) an eigenvector with eigenvalue λ.
   (**c) Let g be a polynomial, and let $\underline{v}$ be the vector with entries $v_k = \lambda^k g(k)$ for $0 \le k \le n-1$. Show that, if the degree of g is small enough (depending on p, λ), then $((C-\lambda)\underline{v})_k = \lambda\,(g(k+1) - g(k))\,\lambda^k$ and (the hard part) that
   $$((C-\lambda)\underline{v})_{n-1} = \lambda\,(g(n) - g(n-1))\,\lambda^{n-1}.$$
   (**d) Find the Jordan canonical form of C.

Holomorphic calculus

Let $f(z) = \sum_{m=0}^{\infty} a_m z^m$ be a power series with radius of convergence R. For a matrix A define $f(A) = \sum_{m=0}^{\infty} a_m A^m$ if the series converges absolutely in some matrix norm.

5. Let $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ be diagonal with $\rho(D) < R$ (that is, $|\lambda_i| < R$ for each i). Show that $f(D) = \operatorname{diag}(f(\lambda_1), \cdots, f(\lambda_n))$.

6. Let $A \in M_n(\mathbb{C})$ be a matrix with $\rho(A) < R$.
   (a) [review of power series] Choose R' such that $\rho(A) < R' < R$. Show that $|a_m| \le C(R')^{-m}$ for some $C > 0$.
   (b) Using PS8 problem 3(a) show that $f(A)$ converges absolutely with respect to any matrix norm.
   (*c) Suppose that $A = S(D+N)S^{-1}$ where $D+N$ is the Jordan form (D is diagonal, N upper-triangular nilpotent). Show that
   $$f(A) = S\left(\sum_{k=0}^{n} \frac{f^{(k)}(D)}{k!} N^k\right) S^{-1}.$$
   Hint: D, N commute.
   RMK1 This gives an alternative proof that $f(A)$ converges absolutely if $\rho(A) < R$, using the fact that $f^{(k)}(D)$ can be analyzed using single-variable methods.
   RMK2 Compare your answer with the Taylor expansion $f(x+y) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} y^k$.
   (d) Apply this formula to find $\exp(tB)$ where B is as in problem 2.