Math 412: Problem Set 6 (due 28/2/2014)

P1. (Minimal polynomials)

(a) Find the minimal polynomial of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(b) Show that the minimal polynomial of $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$ is $x^2(x-1)^2$.

(c) Find a 3×3 matrix whose minimal polynomial is x^2.

P2. (Generalized eigenspaces) Let A be as in P1(b).

(a) What are the eigenvalues of A?

(b) Find the generalized eigenspaces.

The minimal polynomial

1. Let $D \in M_n(F) = \operatorname{diag}(a_1, \ldots, a_n)$ be diagonal.

(a) For any polynomial $p \in F[x]$ show that $p(D) = \operatorname{diag}(p(a_1), \ldots, p(a_n))$.

(b) Show that the minimal polynomial of D is $m_D(x) = \prod_{j=1}^{r}(x - a_{i_j})$ where $\{a_{i_j}\}_{j=1}^{r}$ is an enumeration of the distinct values among the a_i.

(c) Show that (over any field) the matrix of P1(b) is not similar to a digaonal matrix.

(d) Now suppose that U is an upper-triangular matrix with diagonal D. Show that for any $p \in F[x]$, $p(U)$ has diagonal $p(D)$. In particular, $m_D | m_U$.

2. Let $T \in \operatorname{End}(V)$ be diagonable. Show that every generalized eigenspace is simply an eigenspace.

3. Let $S \in \operatorname{End}(U)$, $T \in \operatorname{End}(V)$. Let $S \oplus T \in \operatorname{End}(U \oplus V)$ be the "block-diagonal map".

(a) For $f \in F[x]$ show that $f(S \oplus T) = f(S) \oplus f(T)$.

(b) Show that $m_{T \oplus S} = \operatorname{lcm}(m_S, m_T)$ ("least common multiple": the polynomial of smallest degree which is a multiple of both).

(c) Conclude that $\operatorname{Spec}_F(S \oplus T) = \operatorname{Spec}_F(S) \cup \operatorname{Spec}_F(T)$.

RMK See also problem A below.

Supplementary problems

A. Let $R \in \text{End}(U \oplus V)$ be "block-upper-triangular", in that $R(U) \subset U$.
   (a) Define a "quotient linear map" $\bar{R} \in \text{End}(U \oplus V/U)$.
   (b) Let S be the restriction of R to U. Show that both m_S, $m_{\bar{R}}$ divide m_R.
   (c) Let $f = \text{lcm}[m_S, m_{\bar{R}}]$ and set $T = f(R)$. Show that $T(U) = \{\underline{0}\}$ and that $T(V) \subset U$.
   (d) Show that $T^2 = 0$ and conclude that $f \mid m_R \mid f^2$.
   (e) Show that $\text{Spec}_F(R) = \text{Spec}_F(S) \cup \text{Spec}_F(\bar{R})$.

B. Let $T \in \text{End}(V)$. For monic irreducible $p \in F[x]$ define $V_p = \left\{\underline{v} \in V \mid \exists k : p(T)^k \underline{v} = \underline{0}\right\}$.
   (a) Show that V_p is a T-invariant subspace of V and that $m_{T \restriction V_p} = p^k$ for some $k \geq 0$, with $k \geq 1$ iff $V_p \neq \{\underline{0}\}$. Conclude that $p^k | m_T$.
   (b) Show that if $\{p_i\}_{i=1}^r \subset F[x]$ are distinct irreducibles then the sum $\bigoplus_{i=1}^r V_{p_i}$ is direct.
   (c) Let $\{p_i\}_{i=1}^r \subset F[x]$ be the prime factors of $m_T(x)$. Show that $V = \bigoplus_{i=1}^r V_{p_i}$.
   (d) Suppose that $m_T(x) = \prod_{i=1}^r p_i^{k_i}(x)$ is the prime factorization of the minimal polynomial. Show that $V_{p_i} = \text{Ker}\, p_i^{k_i}(T)$.