MATH 253 – WORKSHEET 21
ITERATED INTEGRALS ON PLANAR DOMAINS

(1) Consider $f(x,y) = \left(1-x^2\right)^{3/2}$ on $D = \left\{x^2+y^2 \le 1\right\}$.
  (a) What is the range of x values in the domain?
  Solution: $x \in [-1,1]$.
  (b) For each x value, what is the range of y values?
  Solution 1: $x^2+y^2 \le 1 \iff y^2 \le 1-x^2 \iff |y| \le \sqrt{1-x^2} \iff -\sqrt{1-x^2} \le y \le +\sqrt{1-x^2}$.
  Solution 2: The range of y values is determined by the intersection points of the vertical line through $(x,0)$ with the unit circle, that is at the points (x,y) where $x^2+y^2=1$, that is at the points (x,y) where $y = \pm\sqrt{1-x^2}$.
  (c) Write the domain in the suggested form.
  Solution: $D = \left\{(x,y) \mid -1 \le x \le 1,\, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\right\}$.
  (d) Set up an iterated integral.
  Solution:

$$\iint_D f\,\mathrm{d}x\,\mathrm{d}y = \int_{x=-1}^{x=+1} \mathrm{d}x \int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} \mathrm{d}y \left(1-x^2\right)^{3/2}$$

  (e) Do the integral.
  Solution:

$$\begin{aligned}
\int_{x=-1}^{x=+1} \mathrm{d}x \int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} \mathrm{d}y \left(1-x^2\right)^{3/2} &= \int_{x=-1}^{x=+1} \mathrm{d}x \left(1-x^2\right)^{3/2} \int_{y=-\sqrt{1-x^2}}^{y=+\sqrt{1-x^2}} \mathrm{d}y \\
&= \int_{x=-1}^{x=+1} \mathrm{d}x \left(1-x^2\right)^{3/2} \left(2\sqrt{1-x^2}\right) \\
&= 2\int_{x=-1}^{x=+1} \mathrm{d}x \left(1-x^2\right)^{4/2} \\
&= 4\int_{x=0}^{x=1} \mathrm{d}x \left(1-x^2\right)^{2} \\
&= 4\int_{x=0}^{x=1} \mathrm{d}x \left(x^4-2x^2+1\right) \\
&= 4\left(\frac{1}{5}-\frac{2}{3}+1\right) = \frac{32}{15} = 2\frac{2}{15},
\end{aligned}$$

  where we have used that $\left(1-x^2\right)^2$ is even.

(2) Let D be the finite region bounded by the curves $x=y$ and $x=2-y^2$. Find $\iint_D y\,\mathrm{d}x\,\mathrm{d}y$.
  Solution: The points of intersection of the two curves have y such that $y = x = 2-y^2$ so $y^2+y-2=0$ or $(y+2)(y-1)=0$. It follows that the intersection points are $(-2,-2)$ and $(1,1)$. Slicing the domain horizontally, we can write it as $D = \left\{(x,y) \mid -2 \le y \le 1,\, 2-y^2 \le x \le y\right\}$. It

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follows that

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\begin{aligned}
\iint\limits_D y\,\mathrm{d}x\,\mathrm{d}y &= \int_{y=-2}^{y=1} \mathrm{d}y y \int_{x=2-y^2}^{x=y} \mathrm{d}x \\
&= \int_{y=-2}^{y=1} \mathrm{d}y y \left(y + y^2 - 2\right) \\
&= \left[\frac{y^3}{3} + \frac{y^4}{4} - y^2\right]_{y=-2}^{y=1} \\
&= \left[\frac{1}{3} + \frac{1}{4} - 1\right] - \left[-\frac{8}{3} + 4 - 4\right] \\
&= 2\frac{1}{4}\,.
\end{aligned}
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