MATH 253 – WORKSHEET 18
MORE LAGRANGE MULTIPLIERS

(1) A large rectangular box without lid is made with 12m^2 of wood. What is the largest possible volume?

Solution: Parametrize the problem by having the sides of the box be of length x, y, z with z the "height". Then the volume of the box is $V(x,y,z) = xyz$ and its area is $A(x,y,z) = 2xz + 2yz + xy$ (no lid!). We thus need to maximize $V(x,y,z)$ subject to $A(x,y,z) = 12$. We first investigate the boundary. If, say, $x \to \infty$ then $xy, 2xz \leq 12$ so $y, z \leq \frac{12}{x}$. It follows that $V(x,y,z) \leq x \cdot \frac{12}{x} \cdot \frac{12}{x} = 12^2 \frac{1}{x}$, and in particular that $V(x,y,z) \to 0$ if $x \to \infty$. If $x = 0$ then $V = 0$. Similar analysis applies if $y \to \infty$ or $z \to \infty$. Thus the maximum must occur in the interior of the domain, and hence be at a point detectable by the Method of Lagrange Multipliers, that is at a point (x,y,z) satisfying

$$\begin{cases} yz & = \lambda\,(2z + y) \\ xz & = \lambda\,(2z + x) \\ xy & = \lambda\,(2x + 2y) \\ A(x,y,z) & = 12 \end{cases}.$$

Subtracting the first two equations we find $(y - x)\,z = \lambda\,(y - x)$. Suppose first that $y \neq x$, so that$z = \lambda$. Then the first equation would read $yz = z\,(2z + y)$. Subtracting yz from that gives $2z^2 = 0$, that is $z = 0$ and $V = 0$, so this possibility does not give the maximum. At the maximum we must therefore have $x = y$. Plugging this into the third equation gives $x^2 = 4\lambda x$ and therefore ($x \neq 0$ for the same reason that $z \neq 0$) $\lambda = \frac{x}{4}$. Plugging this into hte second equation we get $xz = \frac{x}{4}\,(2z + x)$ and dividing by x and solving for z we get $z = \frac{x}{2}$. To conclude: at the maximum we must have $x = y$ and $2z = x$. Now plugging all of this into the constraint equation we get

$$2xz + 2yz + xy = x^2 + x^2 + x^2 = 12$$

which means $x = 2\text{m}$, and hence $y = 2\text{m}$, $z = 1\text{m}$ and $V = 4\text{m}^3$. Since this is the unique solution which could possibly be a maximum (all other solutions had $x = 0$ or $y = 0$ or $z = 0$) this must be the maximum.

(2) Find the angle between the plane $x + 2y + z = 4$ and the line parametrized by $\left(\frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right) + t\,\langle -1, 1, 2\rangle$ where $\left(\frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right)$ lies on the plane.

Solution: Let $P = \left(\frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right)$ and let $X = (x,y,z)$ be an arbitrary point in the plane. Then the angle between the the given line and the line PX satisfies

$$\cos\theta = \frac{\overrightarrow{PX} \cdot \langle -1, 1, 2\rangle}{\left|\overrightarrow{PX}\right| \sqrt{6}}.$$

The angle between the line and the plane is the smallest possible as we vary X, so we look for the largest value of $\cos\theta$ as we vary X. We write the objective function explicitely:

$$C(x,y,z) = \sqrt{6}\cos\theta = \frac{-\left(x - \frac{4}{3}\right) + \left(y - \frac{2}{3}\right) + 2\left(z - \frac{4}{3}\right)}{\sqrt{\left(x - \frac{4}{3}\right)^2 + \left(y - \frac{2}{3}\right)^2 + \left(z - \frac{4}{3}\right)^2}} = \frac{-x + y + 2z - 2}{\Delta},$$

with the shorthand $\Delta = \sqrt{\left(x - \frac{4}{3}\right)^2 + \left(y - \frac{2}{3}\right)^2 + \left(z - \frac{4}{3}\right)^2}$, and the constraint as $F(x,y,z) = x + 2y + z = 4$. Noting that the angle does not depend on the distance of X from P, we see that we might as well optimize on the circle $\left|\overrightarrow{PX}\right| = 1$, so we are optimizing on a closed and bounded set

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and there will be a maximum detectable by differentiation (the method of Lagrange Multipliers). , the resulting system of equations is

$$\begin{cases} -\frac{1}{\Delta} - \frac{(y+2z-x-2)\left(x-\frac{4}{3}\right)}{\Delta^3} & = \lambda \cdot 1 \\ \frac{1}{\Delta} - \frac{(y+2z-x-2)\left(y-\frac{2}{3}\right)}{\Delta^3} & = \lambda \cdot 2 \\ \frac{2}{\Delta} - \frac{(y+2z-x-2)\left(z-\frac{4}{3}\right)}{\Delta^3} & = \lambda \\ x+2y+z & = 4 \end{cases}$$

Multiplying the second equation by 2 and summing the first three equations we find

$$\frac{-1+2\cdot 1+2}{\Delta} - \frac{(y+2z-x-2)\left[\left(x-\frac{4}{3}\right)+2\left(z-\frac{2}{3}\right)+\left(z-\frac{4}{3}\right)\right]}{\Delta^3} = \lambda + 4\lambda + \lambda\,.$$

Since $x+2y+z-4=0$ we conclude that $6\lambda = \frac{3}{\Delta}$ so $\lambda\Delta = \frac{1}{2}$. Multiplying the second equation by Δ we find

$$1 - \frac{(y+2z-x-2)\left(y-\frac{2}{3}\right)}{\Delta^2} = 1$$

and hence either $y=\frac{2}{3}$ or $y+2z-x-2=0$. The second case is impossible (it would force at the same time $-\frac{1}{\Delta}=\lambda$ and $\frac{1}{\Delta}=2\lambda$), so at the maximum (and minimum) we have $y=\frac{2}{3}$. Multiplying the first equation by Δ we get

$$-1 - \frac{\left(\frac{2}{3}+2z-x-2\right)\left(x-\frac{4}{3}\right)}{\left(x-\frac{4}{3}\right)^2+\left(z-\frac{4}{3}\right)^2} = \frac{1}{2}\,,$$

which we can rewrite as

$$\frac{3}{2}\left(\left(x-\frac{4}{3}\right)^2+\left(z-\frac{4}{3}\right)^2\right) = \left(x-\frac{4}{3}\right)\left(\left(\frac{4}{3}+x-2z\right)\right)\,.$$

From the last equation we also have $x+z=4-\frac{4}{3}=\frac{8}{3}$, so $z=\frac{8}{3}-x$ and hence $z-\frac{4}{3}=\frac{4}{3}-x$. The above equation is therefore

$$3\left(x-\frac{4}{3}\right)^2 = \left(x-\frac{4}{3}\right)\left(3x-\frac{12}{3}\right)$$

which is always satisfied. The same holds for the third equation, so we conclude that the maximum and mimimum occur along the line where $y=\frac{2}{3}$, $z=\frac{8}{3}-x$ (we do not expect a unique solution since the angle does not depend on the distance to P). We now calculate the angle itself along this line. We have

$$\cos\theta = \frac{1}{\sqrt{6}}\frac{\left(\frac{12}{3}-3x\right)}{\sqrt{\left(x-\frac{4}{3}\right)^2+\left(\frac{4}{3}-x\right)^2}} = -\frac{3}{\sqrt{6}\sqrt{2}}\frac{\left(x-\frac{4}{3}\right)}{\left|x-\frac{4}{3}\right|} = \begin{cases} -\frac{\sqrt{3}}{2} & x>\frac{4}{3} \\ \frac{\sqrt{3}}{2} & x<\frac{4}{3} \end{cases}.$$

We conlcude that the angle θ varies between a minimum of $\arccos\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$ and a maximum of $\arccos\left(-\frac{\sqrt{3}}{2}\right)=\frac{2\pi}{3}$. In particular the angle between the line and the plane is $\frac{\pi}{3}$.

Remark: This solution is far more complicated than just calculating the angle between the line and the vector normal to the plane. The above discussion is intended to illustrate the error of just taking the angle between the line and a "random" vector in the plane, and to illustrate the method of Lagrange Multipliers.

(3) The plane $x+y+2z=2$ intersects the paraboloid $z=x^2+y^2$ in an ellipse. Find the points on this ellipse closest nearest and farthest from the origin.

Solution: Parametrize points on the paraboloid by their (x,y) coordinates as $\left(x,y,x^2+y^2\right)$. Such a point is on the ellipse iff it is on the plane, that is iff

$$C(x,y) = x+y+2(x^2+y^2) = 2\,.$$

The distance to the origin is given by $\Delta = \sqrt{x^2+y^2+z^2}$, so to find its maximum and minimum it is enough to optimize the squared distance

$$D(x,y,z) = \Delta^2 = x^2+y^2+z^2 = x^2+y^2+\left(x^2+y^2\right)^2 \ .$$

The ellipse is closed and bounded, and has no boundary, so the maximum and minimum must be detectable by the Method of Lagrange Multipliers, that is be at a point (x,y) where

$$\begin{cases} 2x+2\left(x^2+y^2\right)(2x) & = \lambda\,(1+4x) \\ 2y+2\left(x^2+y^2\right)(2y) & = \lambda\,(1+4y) \\ C(x,y) & = 2 \end{cases} ,$$

that is

$$\begin{cases} \left(x^2+y^2+1\right)x & = \frac{\lambda}{2}\,(4x+1) \\ \left(x^2+y^2+1\right)y & = \frac{\lambda}{2}\,(4y+1) \\ C(x,y) & = 2 \end{cases} .$$

Subtracting the first two equations we find

$$\left(x^2+y^2+1\right)(x-y) = 2\lambda\,(x-y) \ .$$

If $x \neq y$ we must have $2\lambda = x^2+y^2+1$ and can therefore divide the first equation by the positive quantity$\left(x^2+y^2+1\right)$ to find

$$\begin{aligned} x &= \frac{\lambda}{2\left(x^2+y^2+1\right)}\,(4x+1) \\ &= \frac{1}{4}\,(4x+1) \\ &= x+\frac{1}{4}\,, \end{aligned}$$

which is impossible. We conclude that $x=y$. Applying this to the constraint we find that the maximum and minimum must occur at a point (x,x) such that

$$2x+2\left(2x^2\right) = 2\,,$$

that is

$$2x^2+x-1 = 0\,.$$

By the quadratic formula the solutions to this are

$$y = x = \frac{-1\pm\sqrt{1+8}}{4} = \frac{-1\pm 3}{4} = \frac{1}{2}, -1\,.$$

In other words, the maximum and minimum must occur among the points $(-1,-1,2)$ and $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ (recall that $z=x^2+y^2$). Since

$$\begin{aligned} D\,(-1,-1,2) &= 6 \\ D\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) &= \frac{3}{4} \end{aligned}$$

we conlcude that the largest distance is $\sqrt{6}$, occuring at $(-1,-1,2)$ and the smallest distance is $\frac{\sqrt{3}}{2}$, occuring at $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$.