MATH 100 – WORKSHEET 30
L'HÔPITAL'S RULE

1. Statement

(1) Evaluate $\lim_{x\to1}\frac{\ln x}{x-1}$

Solution: $\lim_{x\to1}\ln x = \ln_{x\to1}(x-1) = 0$ so we can apply l'Hôpital's rule to get $\lim_{x\to1}\frac{\ln x}{x-1} = \lim_{x\to1}\frac{1/x}{1} = 1$.

Note: Also $\lim_{x\to1}\frac{\ln x}{x-1} = \lim_{x\to1}\frac{\ln x-\ln 1}{x-1} = (\ln x)' \restriction_{x=1} = \left(\frac{1}{x}\right)\restriction_{x=1} = 1$.

(2) Evaluate $\lim_{x\to0}\frac{\cos x-1}{x^2} = \lim_{x\to0}\frac{-\sin x}{2x} = \lim_{x\to0}\frac{-\cos x}{2} = -\frac{\cos 0}{2} = -\frac{1}{2}$.

Solution: We apply l'Hôpital's rule twice, noting that $\lim_{x\to0}x^2 = \lim_{x\to0}2x = \lim_{x\to0}(\cos x - 1) = \lim_{x\to0}\sin x = 0$.

(3) Do (2) using a 2nd-order Taylor expansion.

Solution: We have $\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots$ so $\frac{\cos x-1}{x^2} \approx -\frac{1}{2} + \frac{1}{24}x^2 - \cdots$ so the limit at $x=0$ is $-\frac{1}{2}$.

(4) Given that $f(2)=5$, $g(2)=3$, $f'(2)=7$ and $g'(2)=4$ find $\lim_{x\to3}\frac{f(2x-4)-g(x-1)}{g(x^2-7)}$.

Solution: The numerator has the limit $f(2)-g(2)=2$, the denominator has limit $g(2)=3$ so the limit is $\frac{2}{3}$.

Note: For $\lim_{x\to3}\frac{f(2x-4)-g(x-1)-2}{g(x^2-7)-3}$ both numerator and denominator tend to zero, so we apply l'Hôpital's rule to get

$$\lim_{x\to3}\frac{f(2x-4)-g(x-1)-2}{g(x^2-7)-3} = \lim_{x\to3}\frac{2f'(2x-4)-g'(x-1)}{2x\cdot g'(x^2-7)} = \frac{2f'(2)-g'(2)}{2\cdot3\cdot g'(2)} = \frac{14-4}{6\cdot4} = \frac{5}{12}\,.$$

(5) Evaluate $\lim_{x\to0}\frac{e^x}{x}$.

Solution: $\lim_{x\to0}e^x = 1$ while $\lim_{x\to0}x=0$ so the limit <u>*does not exist*</u>.

(6) Evaluate $\lim_{x\to\infty}x^2e^{-x}$.

Solution: We use l'Hôpital's rule twice, since $\lim_{x\to\infty}x^2 = \lim_{x\to\infty}2x = \lim_{x\to\infty}e^x = \infty$, $\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0$.

(7) Evaluate $\lim_{x\to0}x\ln x$.

Solution: Write as a ratio, putting $\ln x$ in denominator so it'll be hit by a differentation and get simplified:

$$\lim_{x\to0}x\ln x = \lim_{x\to0}\frac{\ln x}{1/x} = \lim_{x\to0}\frac{1/x}{-1/x^2} = \lim_{x\to0}(-x) = 0$$

(8) Evaluate $\lim_{x\to\infty}x^ne^{-x}$.

Solution: applying l'Hôpital's rule n times, we see:

$$\lim_{x\to\infty}\frac{x^n}{e^x} = \lim_{x\to\infty}\frac{nx^{n-1}}{e^x} = \lim_{x\to\infty}\frac{n(n-1)x^{n-2}}{e^x} = \cdots = \lim_{x\to\infty}\frac{n(n-1)\cdots1}{e^x} = n!\lim_{x\to\infty}e^{-x} = 0\,.$$

(9) Suppose $a>0$. Evaluate $\lim_{x\to\infty}x^{-a}\ln x$.

Solution: $\lim_{x\to\infty}\frac{\ln x}{x^a} = \lim_{x\to\infty}\frac{1/x}{ax^{a-1}} = \frac{1}{a}\lim_{x\to\infty}x^{-a} = 0$.

(10) Evaluate $\lim_{x\to0}(2x+1)^{1/\sin x}$.

Solution: We use logarithms to convert exponentiation to multiplication:

$$(2x+1)^{1/\sin x} = \left(e^{\ln(2x+1)}\right)^{1/\sin x} = e^{\frac{\ln(2x+1)}{\sin x}}\,.$$

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We can now apply l'Hôpital's rule:

$$\lim_{x\to 0}\frac{\ln(2x+1)}{\sin x}=\lim_{x\to 0}\frac{2/(2x+1)}{\cos x}=\frac{2/1}{\cos 0}=2\,.$$

By the continuity of e^u we now have

$$\lim_{x\to 0}(2x+1)^{1/\sin x}=\lim_{x\to 0}e^{\frac{\ln(2x+1)}{\sin x}}=e^{\lim_{x\to 0}\frac{\ln(2x+1)}{\sin x}}=e^2\,.$$