MATH 100 – WORKSHEET 22
ESTIMATES ON TAYLOR EXPANSIONS

The Taylor expansion of $f(x)$ about $x = a$ is

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Then there is c between a and x such that

$$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Moral: The remainder looks like the next term except the derivative is evaluated at the point c.

1. Linear approximation of $(1001)^{1/3}$

(1) Estimate $(1001)^{1/3}$ using a linear approximation. Express your answer as a rational number.
(2) Write down the remainder term as it applies to this case. In which range does c vary?
(3) Give an upper bound for the error in your approximation.

Solution. Let $f(x) = x^{1/3}$ so that $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = -\frac{2}{9}x^{-5/3}$.

(1) We have $f(1000) = 10$, $f'(1000) = \frac{1}{300}$ so $f(1001) \approx f(1000) + (1001-1000)f'(1000) = 10 + \frac{1}{300} = \boxed{10\frac{1}{300}}$.

(2) We need $R_1(x) = \frac{f''(c)}{2!}(1001-1000)^2 = \boxed{-\frac{1}{9}c^{-5/3}}$ where c varies between $1000, 1001$.

(3) We see that R_1 is negative, and that its magnitude is decreasing in c, so $\boxed{\left|-\frac{1}{9}c^{-5/3}\right| \leq \frac{1}{9}(1000)^{5/3} = \frac{1}{9\cdot 10^5}}$.

2. Taylor expansion of e^x

Let $f(x) = e^x$ and recall that the Maclaurin expansion is $T_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$.

(1) Estimate e using a second order Taylor expansion. Write your answer as a rational number.
(2) Estimate the error.
(3) Repeat for $\frac{1}{e}$.

Solution. Here we use $f(x) = e^x$ and all derivatives take the same form. We are expanding about $a = 0$.

(1) We have $T_2(x) = 1 + x + \frac{x^2}{2}$ so $e = f(1) \approx 1 + 1 + \frac{1}{2} = \boxed{2\frac{1}{2}}$.

(2) We have $R_2(1) = \frac{f^{(3)}(c)}{3!}1^3 = \frac{e^c}{6}$ for some $0 < c < 1$. We see that $R_2(1) > 0$ and since e^c is increasing here, that $\boxed{0 < R_2(1) < \frac{e^1}{6} = \frac{e}{6}}$, which is at most $\frac{1}{2}$ since $e < 3$ (see below).

(3) First, $\frac{1}{e} = e^{-1} = f(-1) \approx T_2(-1) = 1 + (-1) + \frac{(-1)^2}{2} = \boxed{\frac{1}{2}}$. Second, the error has the form $R_2(-1) = \frac{e^c}{3!}(-1)^3 = -\frac{e^c}{6}$ for some $-1 < c < 0$. We see that $R_2(-1) < 0$ and that the magnitue is at most that when $c = 0$ so $0 > R_2(-1) > -\frac{1}{6}$.

Remark. We get that $\frac{1}{2} - \frac{1}{6} < \frac{1}{e} < \frac{1}{2}$. Since $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$ this reads $\frac{1}{3} < \frac{1}{e} < \frac{1}{2}$ that is $2 < e < 3$.

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3. Taylor expansion of $\sqrt{x}$ about $x = 4$

Let $f(x) = \sqrt{x}$ and recall that about $a = 4$ we have $T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$ and that $f^{(3)}(x) = \frac{3}{8x^{5/2}}$.

(1) Approximate $\sqrt{5}$ using a 2nd order expansion.
(2) Bound the error in your expansion.
(3) Approximate $\sqrt{5}$ using a 3rd order expansion.
(4) Bound the error in your approximation.

Solution.

(1) $T_2(5) = 2 + \frac{1}{4}(1) - \frac{1}{64}(1)^2 = \boxed{2\frac{15}{64}}$.
(2) $R_2(5) = \frac{1}{6} \cdot \frac{3}{8} c^{-5/2}(1)^3$ for $4 < c < 5$. This is positive and at most $\frac{1}{6}\frac{3}{8}(4)^{-5/2}(1)$ since $c^{-5/2}$ is decreasing. We get that $0 < R_2(5) < \frac{1}{512}$.
(3) $T_3(5) = T_2(5) + \frac{1}{512}(1)^3 = \boxed{2\frac{121}{512}}$.
(4) We have $f^{(4)}(x) = -\frac{5}{2}\frac{3}{8}x^{-7/2}$ and hence $R_3(5) = -\frac{15}{16}\frac{c^{-7/2}}{4!}(5-4)^4 = -\frac{5}{128}c^{-7/2}$ for some $4 < c < 5$. It follows that $R_3(5) < 0$ and (since $c^{-7/2}$ is decreasing in c) that $|R_3(5)| < \frac{5}{128}4^{-7/2} = \frac{5}{2^{14}}$.