MATH 100 – WORKSHEET 20
RELATED RATES

1. Related rates

(1) Two ships are travelling near an island. The first is located 20km due west of it and is moving due north at 5km/h. The second is located 15km due south of it and is moving due south at 7km/h. How fast is the distance between the ships changing?

Solution: Place a co-ordinate system with the x-axis running EW, y-axis running NS and with the origin at the island. Measure distances in kilomoteres and time in hours. Then the position of the first ship is $(-20, x(t))$, of the second ship $(0, y(t))$ for some unkown functions x, y. We are given that $x(0) = 0$ (due West of island) and $y(0) = -15$ (due south of the island). We are also given that $x'(t) = 5$ and $y'(t) = -15$. Finally, the distance D between the two ships satisfies:

$$D(t)^2 = (-20 - 0)^2 + (x(t) - y(t))^2 \,.$$

Differentiating we find

$$2D \cdot \frac{dD}{dt} = 2\,(x - y)\left(\frac{dx}{dt} - \frac{dy}{dt}\right) \,.$$

Now at the given time we have $D^2 = 400 + 15^2 = 625$ so $D = 25$ and we thus have:

$$\frac{dD}{dt} = \frac{0 - (-15)}{25}\,(5 - (-15))$$

so

$$\boxed{\frac{dD}{dt} = \frac{15 \cdot 20}{25} = 12\frac{\text{km}}{\text{h}}}$$

(2) The same setting, but now the first ship is moving toward the island.

Now the positions are $(x(t), 0)$ and $(0, y(t))$ with $x(0) = -20$, $y(0) = -15$ and $x'(0) = 5$, $y'(t) = -7$. Now

$$D^2 = x^2 + y^2$$

so

$$2DD' = 2xx' + 2yy' \,.$$

We still have $D = 25$ at the initial time, at which point

$$\boxed{D' = \frac{(-20) \cdot 5 + (-15)(-7)}{25} = \frac{5}{25} = \frac{1}{5}\frac{\text{km}}{\text{h}} \,.}$$

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(3) A conical drain is 6m tall and has radius 1m at the top.

(a) The drain is clogged, and is filling up with rain water at the rate of $5\text{m}^3/\text{min}$. How fast is the water rising when its height is 5m?

Solution: Suppose the cone has height H and radius R at the top. The water also takes up the shape of an inverted cone, say with height $h(t)$ and radius $r(t)$ at the "base" (surface of the water). Then the volume of the water is $V(t) = \frac{1}{3}\pi r(t)^2 h(t)$. Now the right-angled triangles with sides $r(t), h(t)$ and R, H are similar (take a cross-section of the cone), so

$$\frac{r(t)}{h(t0} = \frac{R}{H}$$

and

$$\boxed{r(t) = \frac{R}{H} h(t)\,.}$$

It follows that

$$V(t) = \frac{1}{3}\pi \frac{R^2}{H^2} (h(t))^3 \,.$$

We now differentiate both sides to see:

$$\frac{dV}{dt} = \frac{\pi R^2}{H^2} (h(t))^2 \frac{dh}{dt}$$

and

$$h'(t) = \frac{H^2}{\pi R^2} \frac{1}{(h(t))^2} V'(t)\,.$$

At the given time we have $H = 6$, $R = 1$, $h = 5$ and $V' = 5$. It follows that

$$h'(t) = \frac{36}{\pi \cdot 25} 5 = \frac{36}{5\pi} \frac{\text{m}}{\text{min}}\,.$$

(b) Once the drain is unclogged the water begins to clear at the rate of $15\text{m}^3/\text{min}$ (but rain is still falling!). At what height is the water falling at the rate of $40m/min$.

Solution: We still have the formula $V' = \frac{\pi R^2}{H^2} (h(t))^2 h'$. We are now given that $V' = -10$ (drain 15 cubic metres per minute, gain 5) and that $h' = -40$ at the units we are working with. We thus have

$$(h(t))^2 = \frac{H^2}{\pi R^2} \frac{V'}{h'} = \frac{36}{\pi} \frac{-10}{-40} = \frac{9}{\pi}\,.$$

It follows that the given situation occurs when

$$\boxed{h} = \sqrt{\frac{9}{\pi}} = \frac{3}{\sqrt{\pi}}\text{m}\,.$$