MATH 100 – WORKSHEET 15
LOGARITHMS AND THEIR DERIVATIVES

1. Logarithms

Summary.

$$\boxed{\log_b(b^x) = b^{\log_b x} = x}$$

$$\boxed{\log_b(xy) = \log_b x + \log_b y} \qquad \boxed{\log_b(x^y) = y\log_b x} \qquad \boxed{\log_b \frac{1}{x} = -\log_b x}$$

Review of calculations.

(1) Simplify the following logarithms
  (a) $\ln\left(e^{10}\right) = \boxed{10}$ by definition.
  (b) [Answer in terms of $\ln 2$]. $\ln(2^{100}) = \boxed{100\ln 2}$.
  Justification: (1) $\ln(x^y) = y\ln x$ or (2) $e^{\ln 2} = 2$ means $e^{100\ln 2} = \left(e^{\ln 2}\right)^{100} = 2^{100}$.

(2) A drug in a patient has a metabolic half-life of 6 hours. Suppose a patient ingests a dose D_0 of the drug. Write a formula for the amount of drug present in the patient t hours afterward:

$$D(t) = D_0 \cdot 2^{-\boxed{(t/6)}}$$

Justification: The dose drops by a factor of 2 every 6 hours, so in t hours there are $t/6$ halvings.

(3) A variant on Moore's Law states that computing power doubles every 18 months. Suppose computers today can do N_0 operations per second.
  (a) Write a formula for the power of computers t years into the future:
    - Computers t years from now will be able to do $N(t)$ operations per second where

$$N(t) = \boxed{N_0 2^{2t/3}}.$$

  Justification: $\frac{2}{3}t$ doublings in t years.

  (b) A computing task would take 10 years for today's computers. Suppose we wait 3 years and then start the computation. When will we have the answer?
  Solution: In 3 years computers will be $2^{2\cdot 3/3} = 4$ times as powerful as today's, so the total wait time is $\boxed{3 + \frac{10}{4} = 5.5}$ years.

  (c) At what time will computers be powerful enough to complete the task in 6 months?
  Solution: Computers t years in the future will compete the task in $10 \cdot 2^{-2t/3}$ years, so we should wait until $10 \cdot 2^{-2t/3} = \frac{1}{2}$. Taking logarithms we find

$$-\log 2 = \log\frac{1}{2} = \log\left(10 \cdot 2^{-2t/3}\right) = \log 10 + \log\left(2^{-2t/3}\right) = \log 10 - \frac{2t}{3}\log 2$$

  so

$$\frac{2t}{3} = \frac{\log 10 + \log 2}{\log 2}$$

  and

$$\boxed{t = \frac{3}{2}\frac{\log 20}{\log 2}}.$$

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$$\boxed{(\ln x)' = \frac{1}{x}} \qquad \boxed{f' = f\,(\ln f)'}$$

Example 1. Differentiate $\ln|x|$.

Solution: For $x > 0$ we have $\ln|x| = x$ so $(\ln|x|)' = (\ln x)' = \frac{1}{x}$. For $x < 0$ we have $\ln|x| = \ln(-x)$ so $(\ln|x|)' = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$ by the chain rule. Conclude that

$$(\ln|x|)' = \frac{1}{x}$$

for all $x \neq 0$.

(1) Differentiate

(a) $f(x) = x^2\ln(1+x^2)$. $f'(x) \overset{\text{pdt}}{=} 2x\ln(1+x^2)+x^2\left(\ln(1+x^2)\right)' \overset{\text{chain}}{=} 2x\ln(1+x^2)+x^2\left(\frac{1}{1+x^2}\cdot(2x)\right) = \boxed{2x\ln(1+x^2)+\frac{2x^3}{1+x^2}}$.

(b) $g(r) = \frac{1}{\ln(\sin r)}$. $g'(r) \overset{\text{qout}}{=} -\frac{1}{(\ln\sin r)^2}\left(\ln(\sin r)\right)' \overset{\text{chain}}{=} -\frac{1}{(\ln\sin r)^2}\frac{1}{\sin r}(\sin r)' \overset{\text{chain}}{=} \boxed{-\frac{1}{(\ln\sin r)^2}\frac{\cos r}{\sin r}}$.

(c) $h(t) = \ln\left(t^2+3t\right)$. $h'(t) \overset{\text{chain}}{=} \boxed{\frac{2t+3}{t^2+3t}}$. But also $\ln(t^2+3t) = \ln(t(t+3)) = \ln t + \ln(t+3)$ so $h'(t) = \boxed{\frac{1}{t}+\frac{1}{t+3}}$.

(d) Find y' if $\ln(x+y) = e^y$.

Solution: Differentiating both sides we see $(\ln(x+y))' \overset{\text{chain}}{=} \frac{1}{x+y}(x+y)' = \frac{1+y'}{x+y}$ and $(e^y)' \overset{\text{chain}}{=} e^y y'$ so

$$\frac{1+y'}{x+y} = e^y y'$$

which we can solve for y':

$$\boxed{y' = \frac{1}{(x+y)e^y - 1}}.$$

(2) Logarithm Laws

(a) Using the chain rule, $\frac{d(\ln(ax))}{dx} = \boxed{\frac{1}{ax}\cdot a = \frac{1}{x}}$.

(b) Simplify $\ln(ax)$ and explain why $(\ln(ax))' = (\ln x)'$. $\ln(ax) = \boxed{\ln a + \ln x}$. Since $\ln a$ is constant, $(\ln(ax))' = 0 + (\ln x)' = \frac{1}{x}$ no matter what a is.

(3) Differentiate

(a) Let $f(x) = \frac{x\cos x}{\sqrt{5+x}}$. Then $\ln f = \ln x + \ln\cos x - \frac{1}{2}\ln(x+5)$ so $(\ln f)' \overset{\text{sum}}{=} (\ln x)' + (\ln\cos x)' - \frac{1}{2}\left(\ln(x+5)\right)' \overset{\text{chain}}{=} \frac{1}{x} - \frac{\sin x}{\cos x} - \frac{1}{2(x+5)}$ so $\boxed{\left(\frac{x\cos x}{\sqrt{5+x}}\right)' = \left(\frac{x\cos x}{\sqrt{5+x}}\right)\left(\frac{1}{x} - \frac{\sin x}{\cos x} - \frac{1}{2(x+5)}\right)}$.

(b) Let $f(x) = x^x$. Then $\ln f = x\ln x$ so $(\ln f)' \overset{\text{pdt}}{=} 1\ln x + x\frac{1}{x} = \ln x + 1$ and $\boxed{(x^x)' = x^x(1+\ln x)}$.

(c) Let $f(x) = (\ln x)^{\cos x}$. Then $\ln f = (\cos x)(\ln\ln x)$ and $(\ln f)' \overset{\text{pdt}}{=} -(\sin x)(\ln\ln x) + \cos x\,(\ln\ln x)' \overset{\text{chain}}{=} -\sin x\ln\ln x + \cos x\frac{1}{\ln x}(\ln x)' = -\sin x\ln\ln x + \frac{\cos x}{x\ln x}$ so

$$\boxed{\left((\ln x)^{\cos x}\right)' =} (\ln x)^{\cos x}\left[\frac{\cos x}{x\ln x} - \sin x\ln\ln x\right].$$