1. Find the absolute maximum and minimum of $f(x) = \ln(x^2 + x + 1)$ in the interval $[-1, 1]$.

   Solution: f is differentiable on the interval ($x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$ so f is well-defined for all x, differentiable by the chain rule, hence continuous) so acheives its minimum and maximum. These must occur at endpoints or critical points. $f'(x) = \frac{1}{x^2+x+1}\left(x^2 + x + 1\right)' = \frac{2x+1}{x^2+x+1}$ and this vanishes only if $2x+1 = 0$ ie if $x = -\frac{1}{2}$. We have $f(-1) = \ln(1 - 1 + 1) = \ln 1 = 0$, $f(1) = \ln 3$ and $f(-\frac{1}{2}) = \ln\left(\frac{1}{4} - \frac{1}{2} + 1\right) = \ln\frac{3}{4}$. Since $\frac{3}{4} < 1 < 3$ we have $\ln\frac{3}{4} < 0 < \ln 3$ so the absolute maximum is $\ln 3$ at $x = 1$ and the absolute minimum is $\ln\frac{3}{4}$ at $x = -\frac{1}{2}$.

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2. The function f is continuous and differentiable on the interval $[2, 5]$. If $f(5) = 7$ and $f'(x) \leq 2$ what is the smallest $f(2)$ can be?

   Solution: Since f is differentiable on $[2, 5]$ we can apply the MVT there to get $c \in (2, 5)$ such that $\frac{f(5)-f(2)}{5-2} = f'(c) \leq 2$. It follows that $f(5) - f(2) \leq 2 \cdot (5 - 2) = 6$ so

$$f(2) \geq f(5) - 6 = 1\,.$$

   [to see that $f(2) = 1$ is possible consider $f(x) = 2x - 3$).

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3. Two runners start a race at the same time, and finish in a tie. Show at the some time during the race they were running at the same speed. (Hint: use the function $h(t) = f(t) - g(t)$ where $f(t), g(t)$ are the position functions of the runners.

   Solution: Let T be the time of the end of the race. We are then given that $h(0) = f(0) - g(0) = 0$ (the runners start together) and $h(T) = f(T) - g(T) = 0$ (they finish together). We suppose that h is differentiable, at which point by the MVT there is $t \in (0, T)$ such that $h'(t) = \frac{h(T)-f(0)}{T-0} = 0$. This means $f'(t) - g'(t) = 0$ so at the time t the two runners are running at the same speed.