1. For what values of c is f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 1 & \text{if } x \leq 3 \\ 2x + c & \text{if x>3} \end{cases}$$

Solution: f is continuous everywhere except 3 (defined by formula without bad points). At 3, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(cx^2+1) = c(3^2)+1 = 9c+1 = f(3)$ and $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(2x+c) = 2\cdot 3 + c$. So both the right- and left- limit exists, and the left one is equal to the value of the function. So f is continuous if and only if the right limit is also equal to this value, that is fi $9c+1 = 6+c$. which is equivalent to $c = \frac{5}{8}$.

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2. Show that for some x we have $f(x) = 100$ if

$$f(x) = x^3 + x \sin x$$

Solution: f is continuous everywhere. Also $f(0) = 0^3 + 0 \sin 0 = 0$ and $f(10) = 1000 + 10 \sin 10 \geq 1000 - 10 = 990$. Since 100 lies between $f(0)$ and $f(10)$, by the IVT there is x between 0 and 10 such that $f(x) = 100$.

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3. Find

$$\lim_{x\to\infty} x\left(\sqrt{x^2+a} - \sqrt{x^2+b}\right)$$

Solution: We have

$$\begin{aligned}
\lim_{x\to\infty} x\left(\sqrt{x^2+a} - \sqrt{x^2+b}\right) &= \lim_{x\to\infty} x\frac{\left(\sqrt{x^2+a} - \sqrt{x^2+b}\right)\left(\sqrt{x^2+a} + \sqrt{x^2+b}\right)}{\left(\sqrt{x^2+a} + \sqrt{x^2+b}\right)} \\
&= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+a} + \sqrt{x^2+b}} \cdot \left((x^2+a) - (x^2+b)\right) \\
&= \lim_{x\to\infty} \frac{a-b}{\sqrt{1+a/x^2} + \sqrt{1+b/x^2}} \\
&= \frac{a-b}{\sqrt{1+a\lim_{x\to\infty}\frac{1}{x^2}} + \sqrt{1+b\lim_{x\to\infty}\frac{1}{x^2}}} \\
&= \frac{a-b}{\sqrt{1+a\cdot 0} + \sqrt{1+a\cdot 0}} = \frac{a-b}{2}.
\end{aligned}$$