Math 613: Problem set 4 (due 18/10/09)

The weight-k action

OPT For a field F let $\mathbb{P}^1(F)$ denote the set of 1-dimensional subspaces of F^2. Write $\begin{bmatrix} a \\ b \end{bmatrix}$ for the subspace generated by the vector $\begin{pmatrix} a \\ b \end{pmatrix} \in F^2 \setminus \{0\}$.

(a) Show that a set of representatives for $\mathbb{P}^1(F)$ is given by $\left\{ \begin{bmatrix} z \\ 1 \end{bmatrix} \right\}_{z \in F}$ togther with the "point at infinity" $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which we can also denote $\begin{bmatrix} \infty \\ 1 \end{bmatrix}$.

(b) Show that the action of $\mathrm{GL}_2(F)$ on F^2 induces an action on $\mathbb{P}^1(F)$, given in co-ordinates by $g \begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{az+b}{cz+d} \\ 1 \end{bmatrix}$ (don't forget the case $z = \infty$).

— Let $j(g,z) = cz + d$ so that $g \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} g \cdot z \\ 1 \end{pmatrix} j(g,z)$. For f defined on $\mathbb{P}^1(F)$ set (formally) $(f|_k g)(z) = f(g \cdot z) j(g,z)^{-k}$.

(c) Show that $j(g \cdot g', z) = j(g, g' \cdot z) j(g', z)$.

(d) Show that $f \mapsto f|_k g$ is a right action of $\mathrm{SL}_2(F)$.

(e) Using $\det \begin{pmatrix} z + dz & z \\ 1 & 1 \end{pmatrix} = dz$ show that $d(gz) = \frac{1}{j(g,z)^2} dz$ as formal differentials on $\mathbb{P}^1(F)$.

2. (Linear independence) We now specialize to the case of $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{PGL}_2(\mathbb{C})$ acting on $\mathbb{P}^1(\mathbb{C})$, where the action restricts to an action on $\mathbb{H}$.

(a) Show that $j(g,z) \neq 0, \infty$ for $g \in \mathrm{SL}_2(\mathbb{R})$, $z \in \mathbb{H}$, so that the formal calculation of part 1 applies here.

(b) Show that $j(g,z) = j(g',z)$ as functions iff $g' g^{-1} \in P$.

(c) Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a discrete subgroup and assume that $\Gamma_\infty = \Gamma \cap P$ is of infinite index in Γ. Find $\{\gamma_m\}_{m=1}^\infty \subset \Gamma$ such that $\{j(\gamma_m, z)\}$ are distinct functions.

(d) Choose $f_k \in \Omega_k(\Gamma)$ for each k (such that all but finitely many are zero) and assume that $\sum_k f_k = 0$. Show that for each m we have $\sum_k j(\gamma_m, z)^k f_k(z) = 0$.

(e) Show that for m large enough the system of linear equations above for f_k is invertible, and conclude that each $f_k = 0$.

(f) Conclude that the sum $\sum_k \Omega_k(\Gamma)(\Gamma)$ is direct.

More on cusps

3. Let Γ be a Fuchsian group of the first kind, $X_\Gamma = \Gamma \backslash \mathbb{H}^*$ its associated closed Riemann surface, $\mathcal{F}_\mathrm{D}$ a Dirichlet fundamental domain. Fix the element $T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$.

(a) Let $\xi, \eta \in \partial\mathbb{H}$ be two vertices at infinity of cFD. Show that they are Γ-inequivalent.
Hint: Wlg $\xi = \infty$, and assume $\gamma\eta = \xi$. Show that there are $w \in \mathcal{F}_\mathrm{D}^\circ$ is close enough to η and $\delta \in \Gamma_\xi$ such that $\delta\gamma w \in \mathcal{F}_\mathrm{D}^\circ$.

— Conclude that the vertices at infinity $\{\xi_k\}_{k=1}^K$ of $\mathcal{F}_{\mathrm{D}}$ are representatives for the Γ−equivalence classes of cusps of Γ.

(b) For each k let $\sigma_k \in \mathrm{SL}_2(\mathbb{R})$ be such that $\sigma_k\infty = \xi_k$. Show that $\sigma_k^{-1}\mathcal{F}_{\mathrm{D}} \cap \{y(z) > Y\}$ is a vertical strip $[x_0, x_0+h] \times (Y,\infty)$ for Y large enough.
Hint: Consider the two sides meeting at the vertex ξ_k.

OPT Show that we can choose σ_k such that $\sigma_k^{-1}\mathcal{F}_{\mathrm{D}} \cap \{y(z) > Y\} = \left[-\frac{1}{2}, \frac{1}{2}\right] \times (Y,\infty)$ and that in that case the image of $\sigma_k^{-1}\Gamma_{\xi_k}\sigma_k$ in $\mathrm{PSL}_2(\mathbb{R})$ is the group generated by T.

(d) Set $\mathcal{F}_{k,Y} = \sigma_k\left[-\frac{1}{2}, \frac{1}{2}\right] \times (Y,\infty)$ and $\mathcal{F}_Y = \mathcal{F}_{\mathrm{D}} \setminus \bigcup_k \mathcal{F}_{k,Y}$. Show that for Y large the $\mathcal{F}_{k,Y}$ are disjoint and $\mathcal{F}_Y$ is compact.

4. The *invariant height* on $\Gamma\backslash\mathbb{H}$ is defined by

$$y_\Gamma(z) = \max_k \max_{\gamma\in\Gamma} y(\sigma_k \gamma z) .$$

(a) Show that $\max_{\gamma\in\Gamma} y(\sigma_k\gamma z)$ is finite and continuous.
Hint: By problem set 3, problem 8(c) the set of y-values is discrete and bounded above.

(b) Show that y_Γ is a continous Γ-invariant function on HH. Show that $y_\Gamma(z_n) \to \infty$ if z_n approach a cusp.

(c) Show that $\{z \in \Gamma\backslash\mathbb{H} \mid y_\Gamma(z) \le Y\}$ is compact, and that if $y_\Gamma(z_n) \to \infty$ then there is a subsequence which converges to a cusp.
Hint: The first part is variant of 3(d).

5. Let $f \in \mathcal{A}_0(\Gamma) = \mathbb{C}(X_\Gamma)$ be a meromorphic function on X_Γ.

(a) Show that for Y large enough f has no zeroes or poles in the region $y_\Gamma(z) > Y$.

— Assume now that Y is also large enough for 3(d) to hold. Let C_Y be the contour that goes along the boundary of $\mathcal{F}_{\mathrm{D}}$ except that at each cusps one truncates the cusp along the curve $y_\Gamma = Y$, and write $C_Y = C_0 \bigcup \cup_k C_k$ where $C_0 = C_Y \cap \partial\mathcal{F}_{\mathrm{D}}$ and C_k is the closed horocycle at the kth cusp.

(b) Show that $\frac{1}{2\pi i}\oint_{C_0} \frac{f'}{f} dz = 0$ using the side-pairings and the invariance of y_Γ.

(c) Evaluate $\frac{1}{2\pi i}\int_{C_k} \frac{f'}{f} dz$ in terms of the behaviour of f at ξ_k by mapping the cusp neighbourhood to a punctured disk.

(d) Since $\frac{1}{2\pi i}\oint_{C_Y} \frac{f'}{f} dz$ counts the zeroes and poles in $\mathcal{F}_Y$, show that f has the same number of zeroes and poles in X_Γ.

6. $X(1) = \Gamma(1)\backslash\mathbb{H}^*$. We have seen in class that $j\colon X(1) \to \mathbb{P}^1(\mathbb{C})$ is a biholomorphism. In particular, all values are simple.

(a) Let $f \in \mathcal{A}_0(\Gamma(1))$ be non-constant. Construct $g \in \mathbb{C}(j)$ such that f, g have the same zeroes and poles in $Y(1)$.
Hint: $j(z) - j(z_0)$ has a simple zero at z_0, a pole at the cusp, and no other zeroes or poles.

(b) Show that $\frac{f}{g}$ has no zeroes or poles in $Y(1)$, and conclude that it has no zeroes or poles in $X(1)$.

(c) Applying the maximum principle show that $\frac{f}{g}$ is constant and conclude that $\mathbb{C}(X(1)) = \mathcal{A}_0(\Gamma(1)) = \mathbb{C}(j)$.

On the choice of σ_ξ

7. Let Γ be a Fuchsian group with a cusp ξ, and let $\sigma, \sigma' \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma\infty = \sigma'\infty = \xi$. Let $f \in \Omega_k(\Gamma)$.
   (a) Show in the definition of f being meromorphic/holomorphic/vanishing at ξ using σ or σ' would not change the conclusion.
   (b) Assume that f is meromorphic at ξ or holomorphic on $\mathbb{H}$. In either case show that the Fourier expansion of f at ξ is essentially independent of the choice σ or σ'. Is the expansion truly independent of the choice?

The cusps of congruence subgroups

8. Let Γ be a Fuchsian group, and let Γ' be a subgroup of finite index.
   (a) Show that Γ and Γ' have the same cusps.
   (b) Let ξ be a cusp of Γ. Show that the Γ'-equivalence classes of cusps which are Γ-equivalent of ξ are in bijection with the double coset space $\Gamma' \backslash \Gamma / \Gamma_\xi$.
   (c) Let $\Gamma_N < \Gamma'$ be normal in Γ, and write bars for the image in the quotient group $\bar{\Gamma} = \Gamma_N \backslash \Gamma$. Show that the map $\Gamma \to \bar{\Gamma}$ induces a bijection $\Gamma' \backslash \Gamma_N / \Gamma_\xi \to \overline{\Gamma'} \backslash \bar{\Gamma} / \overline{\Gamma_\xi}$.

9. Let $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ and recall its subgroups $\Gamma(N) < \Gamma_0(N) < \Gamma_1(N)$ from Problem set 3.
   (a) Show that the cusps of $\Gamma(1)$ are precisely $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\} \subset \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}$, and that $\Gamma(1)$ acts transitively there.
   — Let $\Gamma_\infty = \Gamma(1)_{i\infty}$ and let $\Gamma_\infty^+ = \langle T \rangle$ where T is the translation.
   (b) Let $\bar{\Gamma} = \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Show that the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c,d)$ induces a bijection between $\overline{\Gamma} / \overline{\Gamma_\infty^+}$ and the set of elements of order N in $(\mathbb{Z}/N\mathbb{Z})^2$.
   Hint: This was already done in PS3.
   (b) Show that $X_0(N) = X_{\Gamma_0(N)}$ has $\sum_{d|N} \phi\left((d, N/d)\right)$ cusps. In particular, for p prime $X_0(p)$ has two cusps – in this case find representatives.
   OPT Count the cusps of $X(N) = X_{\Gamma(N)}$ and $X_1(N) = X_{\Gamma_1(N)}$.

Dirichlet characters

Let $N \geq 1$. A *Dirichlet character mod N* is a non-zero function $\chi : \mathbb{Z} \to \mathbb{C}$ such that $\chi(ab) = \chi(a)\chi(b)$, $\chi(a) = \chi(b)$ if $a \equiv b\,(N)$ and $\chi(a) = 0$ whenever $(a,N) > 1$. We freely identify χ with the function it induces on $\mathbb{Z}/N\mathbb{Z}$

OPT. Let χ be a Dirichlet character mod N.
   (a) Show that $\chi(1) = 1$ and that χ is non-zero in $(\mathbb{Z}/N\mathbb{Z})^\times$.
   (b) Show that the non-zero values taken by χ are roots of unity.
   (c) Let $N|M$. Show that the function $\chi_M(a) = \begin{cases} \chi(a) & (a,M) = 1 \\ 0 & (a,M) > 1 \end{cases}$ is a Dirichlet character mod M satisfying $\chi_M(a + kN) = \chi_M(a)$ for all $k \in \mathbb{Z}$. Characters mod M obtained this way with $N < M$ are called *imprimitive*. Other characters are called *primitive*.
   (d) Assume that $\chi_M = \psi_M$ for another character ψ mod N. Show that $\chi = \psi$.