Math 422/501 Problem set 1 (due 16/9/09)

Some group theory

1. (Cyclic groups)
   (a) Show that the infinite cyclic group $\mathbb{Z}$ is the unique group which has non-trivial proper subgroups and is isomorphic to all of them.
   (b) [optional] which groups have no non-trivial proper subgroups?

2. (Groups with many involutions) Let G be a finite group, and let $I = \left\{g \in G \mid g^2 = e\right\} \setminus \{e\}$ be its subset of *involutions* (e is the identity element of G).
   (a) Show that G is abelian if it has *exponent* 2, that is if $G = I \cup \{e\}$.
   (b) Show that G is abelian if $|I| \geq \frac{3}{4}|G|$.

Some polynomial algebra

3. Show that $(x-y)$ divides $(x^n - y^n)$ in $\mathbb{Z}[x,y]$. Conclude that for any ring R, polynomial $P \in R[x]$ and element $a \in R$ such that $P(a) = 0$ one has $(x-a)|P$ in $R[x]$.
4. Let R be an integral domain, $P \in R[x]$, $\{a_i\}_{i=1}^{k} \subset R$ distinct zeroes of P. Show that $\prod_i (x-a_i)|P$ in $R[x]$. Give a counterexample when R has zero-divisors.
5. Let $\mathscr{V}_n(x_1,\ldots,x_n) \in M_n(\mathbb{Z}[x_1,\ldots,x_n])$ be the *Vandermonde matrix* $(\mathscr{V}_n)_{ij} = x_i^{j-1}$. Let $V_n(\underline{x}) = \det(\mathscr{V}_n(\underline{x})) \in \mathbb{Z}[\underline{x}]$. Show that there exists $c_n \in \mathbb{Z}$ so that $V_n(\underline{x}) = c_n \prod_{i>j}(x_i - x_j)$.
   Hint: Consider V_n as an element of $(\mathbb{Z}[x_1,\ldots,x_{n-1}])[x_n]$.
6. Setting $x_n = 0$ show that $c_n = c_{n-1}$, hence that $c_n = 1$ for all n.

Some abstract nonsense

DEFINITION. Let G,H be groups, and let $f\colon G \to H$ be a homomorphism. Say that f is a *monomorphism* if for every group K and every two distinct homomorphisms $g_1, g_2\colon K \to G$, the compositions $f \circ g_1, f \circ g_2\colon K \to H$ are distinct. Say that f is an *epimorphism* if for every group K and every two distinct homomorphisms $g_1, g_2\colon H \to K$ the compositions $g_i \circ f\colon G \to K$ are distinct.

7. Show that a homomorphism of groups is a monomorphism iff it is injective, an epimorphism iff it is surjective.

8. (Variants)
   (a) Same as 7, but replace "group" with "vector space over the field F" and "homomorphism" with "F-linear map".
   (b) Consider now the case of rings and ring homomorphisms. Show that monomorphisms are injective, but show that there exist non-surjective epimorphisms..

*9. Replacing "groups" with "Hausdorff topological spaces" and "homomorphism" with "continuous map" show that:
   (a) A continuous map is a monomorphism iff it is injective.
   (b) A continuous map is an epimorphism iff its image is dense.

CHAPTER 2

Group actions