HOMEWORK 4: MATH 265 Due in class on Oct 20

PARTIAL SOLUTIONS

Problem 1:

(a) Solve the nonhomogeneous ODE $y'' + 16y = \cos(\omega t)$ with $y(0) = 0$, $y'(0) = 0$ and express your solution in terms of the frequency of the forcing term $\omega > 0$. Sketch the solution when $\omega = 3$ and when $\omega = 4$.

(b) **NEW (Can be added at end of your HW pages, if you have already done other problems)** In Part (a), you may have gotten a solution in the form of a sum or difference of two trig functions such as sines or cosines. Use a trigonometric identity to re-express that solution as a product of two trig functions (such as $K\sin(w_1 t)\sin(w_2 t)$ - but you must find the values of K, w_1, w_2 in terms of other quantities in the problem).

(c) Based on part (b) explain the behaviour of the solution when ω is close to but not equal to 4. That is, explain the phenomenon of *beats*. (You may find it useful to read about this in your book.)

(d) Use your favorite graphics software (or calculator) to plot the solution in two cases, $\omega = 3.5$ and $\omega = 3.8$. What is the period of the *envelope* of the oscillations in each case? What is the frequency of the oscillations within that envelope in each case?

Solution to Problem 1:

First solve the homogeneous problem. Let $y = e^{rt}$. Then, the characteristic polynomial is $r^2 + 16 = 0$, so that $r = \pm 4i$. Thus, $y = \sin(4t)$ and $y = \cos(4t)$ are solutions to the homogeneous problem.

• Find the particular solution when $\omega_0 \neq 4$. The particular solution is easily found to be

$$y_p = \frac{\cos(\omega_0 t)}{16 - \omega_0^2}\,.$$

The general solution, when $\omega_0 \neq 4$, is

$$y = c_1\cos(4t) + c_2\sin(4t) + \frac{\cos(\omega_0 t)}{16 - \omega_0^2}\,.$$

Satisfying the initial conditions $y(0) = y'(0) = 0$, we get $c_1 = 0$, $c_2 = -1/(16 - \omega_0^2)$. Thus, the solution to the IVP is

$$y(t) = \frac{1}{(16 - \omega_0^2)}\left[\cos(\omega_0 t) - \cos(4t)\right]\,, \qquad \text{when} \quad \omega_0 \neq 4\,. \tag{2}$$

• When $\omega_0 = 4$ we have resonance and $y_p = (t/8)\sin(4t)$. The general solution is thus,

$$y = c_1\cos(4t) + c_2\sin(4t) + \frac{t}{8}\sin(4t)\,.$$

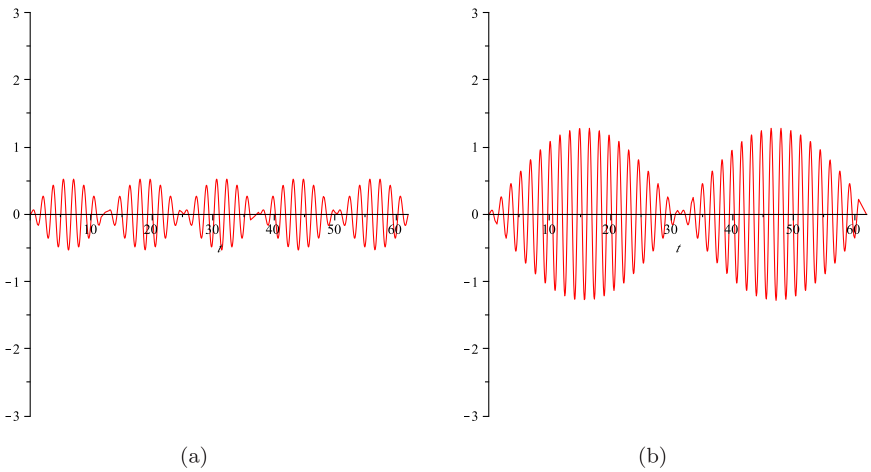

Figure 1: Graphs of the oscillations in Problem 1 for (a) $\omega = 3.5$, (b) $\omega = 3.8$.

Satisfying the initial conditions we get $c_1 = c_2 = 0$. Hence, the solution to the initial value problem is

$$y = \frac{t}{8}\sin(4t)\,, \qquad \text{when} \quad \omega_0 = 4\,. \tag{3}$$

When $\omega_0 \neq 4$, the solution looks like the "beats" shown in Fig 1. When $\omega_0 = 4$ we get an oscillation with increasing amplitude (in fact, the amplitude goes as $t/8$).

Problem 2: Determine the form of the particular solution to the following ODEs and use the method of undetermined coefficients to solve for any constants in that particular solution.

(a) $y'' + y' + y = e^{-t}$

(b) $y'' + y' + y = e^{-t/2}$

(c) $y'' + y' + y = t^2$

(d) $y'' + y = t\cos(t)$

Answers to Problem 2: (The details of the solutions and the algebra involved is not displayed this time. See examples online or in the book if you are uncertain how to find these answers.)

(a) The solution to the homogenous equation is $y(t) = e^{-t/2}\left(c_1\cos(\sqrt{3}t/2) + c_2\sin(\sqrt{3}t/2)\right)$. The particular solution is: $Y_p(t) = e^{-t}$.

(b) This has the same solution to the homogeneous equation. The particular solution is $Y_p(t) = \frac{4}{3}e^{-t/2}$.

(c) It is necessary to assume that the form of the particular solution is $Y_p(t) = At^2 + Bt + C$ and once you solve for the constants you obtain $Y_p(t) = t^2 - 2t$.

(d) In this case, the solution to the homogeneous problem is $y(t) = c_1 \cos(t) + c_2 \sin(t)$ so we would ordinarily guess a particular solution of the form $y_p(t) = (At + B)\cos(t) + (Ct + D)\sin(t)$, but the components $B\cos(t)$ and $D\sin(t)$ duplicate the solution to the homogeneous equation. Thus we need to revise the guess to $y_p(t) = t[(At + B)\cos(t) + (Ct + D)\sin(t))]$ and the particular solution turns out to be $Y_p(t) = \frac{1}{4}t(\cos(t) + t\sin(t))$.

<u>Problem 3:</u> Consider the spring-mass system with forcing, $my'' + \gamma y' + ky = F\cos(wt)$. Note that w is the forcing frequency.

(a) Show that if there is no damping, and w is different from the natural frequency $w_0 = \sqrt{k/m}$, then the general solution is of the form $y(t) = c_1 \cos(w_0 t) + c_2 \sin(w_0 t) + Y_p(t)$, where $Y_p(t) = A\cos(wt) + B\sin(wt)$ is a particular solution. Find the values of A, B.

(b) Now consider the case $\gamma \neq 0$. Find the solution to the homogeneous problem. Suppose the driving frequency is $w = \sqrt{k/m}$. What would be the form of the particular solution? (Do not solve for constants.)

(c) Use a trigonometric identity to show that $Y_p(t) = A\cos(wt) + B\sin(wt) = R\cos(wt - \delta)$, i.e. find R and δ in terms of A, B.

(d) Use a trigonometric identity to show that $\cos(wt) = \cos(\delta)\cos(wt - \delta) - \sin(\delta)\sin(wt - \delta)$

(d) Assume a particular solution in the form $Y_p = R\cos(wt - \delta)$ to the forced damped system. Solve for the constants R and δ in this particular solution. (You are asked to show the algebraic detailed steps, not just write down the answer, which is in your book.) You will have to use part (d) to write the forcing function in terms of $\cos(wt - \delta)$ and $\sin(wt - \delta)$ so as to equate "like terms".

(e) Show the detailed steps in the derivation of the dependence of the oscillation amplitude R on the forcing frequency w, i.e. derive the formula

$$R\frac{k}{F} = \frac{1}{\sqrt{\frac{w^2}{w_0^2}\Gamma + (1 - \frac{w^2}{w_0^2})^2}}$$

where $\Gamma = \gamma^2/(mk)$ is a damping-dependent factor and $w_0 = \sqrt{k/m}$ is the natural frequency of the undamped system. Sketch the quantity $R\frac{k}{F}$ as a function of the driving frequency w, paying particular attention to its value for small and large w and for $w = w_0$. Explain what happens near $w = w_0$ as the damping γ is decreased to very small values.

Solution to Problem 3:

(a) For $\gamma = 0$ the roots of characteristic equation are $r = \pm i\sqrt{k/m}$ so the homogeneous system has the solution $y(t) = c_1 \cos(w_0 t) + c_2 \sin(w_0 t)$. Since the forcing frequency is different from the natural frequency, we take the form of $Y_p(t)$ to be $Y_p(t) = A\cos(wt) + B\sin(wt)$. Solving for A, B leads to $A = F/(k - w^2 m), B = 0$.

(b) $y(t) = e^{\sigma t}(c_1 \cos(w_0 t) + c_2 \sin(w_0 t))$ where $\sigma = -\gamma/2m$. Thus we can use the particular solution $Y_p(t) = A\cos(wt) + B\sin(wt)$, as it does not duplicate any parts of the homogeneous solution.

(b-e) See scanned pages.

Problem 4: Consider an LRC circuit with all elements in series and with time dependent voltage. We showed in class that the charge on the capacitor satisfies a 2nd order ODE as follows: $Lq'' + Rq' + q/C = V(t)$. Now suppose that the applied voltage is periodic, i.e. assume that $V(t) = F\sin(wt)$ where w is the forcing frequency.

(a) Find the corresponding 2nd order ODE for the current I(t) in this circuit. (Recall that $dq/dt = I(t)$). Your ODE should (clearly) be in terms of $I(t)$ only, with no $q(t)$ variable remaining in the equation.

(b) Reinterpret the results you got in Problem 3 for this new problem, that is, find how the frequency and amplitude of current oscillations depends on the driving frequency and on the other parameters L, R, C, F in the problem. NOTE: you do not need to redo any of the algebra or calculations of Problem 3, only to summarize the major findings about the the dependence of the oscillation amplitude R on the forcing frequency w.

Solution to Problem 4: Solution will be available at later date.

Problem 5: Attach 1-2 pages to your homework assignment with corrections to your midterm test. This is an important part of your homework assignment. If you did not yet pick up your test, you can get it in class or in office hours on Monday Oct 18.