HOMEWORK 3 Solutions: Math 265 Leah Keshet
I thank Prof Daniel Coombs for making these available to us for your practice and learning.

Problem 1: Solve the following initial value problems for $y(x)$:
(a) $y'' - 4y' - 5y = 0, \quad y(-1) = 3, \quad y'(-1) = 9.$
(b) $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = 6, \quad y''(0) = 0.$
(Hint: this is a linear equation whose characteristic equation is cubic. Recall that for a cubic equation $r^3 + ar^2 + br + c = 0$ with roots r_1, r_2, r_3, it is true that $c = r_1 r_2 r_3$.)
(c) $y'' + y = 2e^{-x}, \quad y(0) = 0, \quad y'(0) = 0.$
(d) $y'' + 2y' + y = x^2 + 1 - e^x, \quad y(0) = 0, \quad y'(0) = 2.$
(e) $y'' - 2y' + y = 8e^t, \quad y(0) = 3, \quad y'(0) = 2.$
(f) $y'' + 2y' + 2y = 5\cos(2x), \quad y(\pi) = -1/2, \quad y'(\pi) = 1.$

Problem 1 Solution:
(a) The characteristic equation is $r^2 - 4r - 5 = 0$ which has roots $r_1 = -1$ and $r_2 = 5$. The general solution is then $y(t) = C_1 e^{-x} + C_2 e^{5x}$. Applying the initial conditions we find $C_1 = e^{-1}$, $C_2 = 2e^5$, and the solution to the initial value problem: $y(t) = e^{-(x+1)} + 2e^{5(x+1)}$.

(b) The characteristic equation is $r^3 + 2r^2 - 5r - 6 = 0$ which has roots $r_1 = 2$, $r_2 = -1$, and $r_3 = -3$. The general solution is then $y(t) = C_1 e^{2x} + C_2 e^{-x} + C_3 e^{-3x}$. Applying the initial conditions we find $C_1 = 1$, $C_2 = 2$, $C_3 = -1$, and the solution to the initial value problem: $y(t) = e^{2x} + 2e^{-x} - e^{-3x}$.

(c) First consider the homogeneous equation $y_h'' + y_h = 0$. The associated characteristic equation is $r^2 + 1 = 0$ which has roots $r_{1,2} = \pm i$. The homogeneous solution is then $y_h(x) = C_1 \cos(x) + C_2 \sin(x)$. To find the particular solution to the inhomogeneous equation $y_p'' + y_p = 2e^{-x}$ pose the guess $y_p(x) = Ae^{-x}$. Substituting the guess into the equation, noting that $y_p''(x) = Ae^{-x}$, we obtain $Ae^{-x} + Ae^{-x} = 2e^{-x}$, and find that $A = 1$. The particular solution is $y_p(x) = e^{-x}$. The general solution is then $y(x) = y_h(x) + y_p(x)$ $\Rightarrow y(x) = C_1 \cos(x) + C_2 \sin(x) + e^{-x}$. Applying the initial conditions we find $C_1 = -1$, $C_2 = 1$, and the solution to the initial value problem: $y(x) = -\cos(x) + \sin(x) + e^{-x}$.

(d) First consider the homogeneous equation $y_h'' + 2y_h' + y_h = 0$. The associated characteristic equation is $r^2 + 2r + 1 = 0$ which has $r = -1$ as a repeated root. The homogeneous solution is then $y_h(x) = C_1 e^{-x} + C_2 x e^{-x}$. To find the particular solution to the inhomogeneous equation $y_p'' + 2y_p' + y_p = x^2 + 1 - e^x$ pose the guess $y_p(x) = Ax^2 + Bx + C + De^x$. Substituting the guess into the equation, noting that $y_p'(x) = 2Ax + B + De^x$ and $y_p''(x) = 2A + De^x$, we obtain $(2A + De^x) + 2(2Ax + B + De^x) + (Ax^2 + Bx + C + De^x) = x^2 + 1 - e^x$ $\Rightarrow Ax^2 + (4A + B)x + (2A + 2B + C) + 4De^x = x^2 + 1 - e^x$. So $A = 1$, $4A + B = 0$, $2A + 2B + C = 1$, $4D = -1$ and, solving for A, B, C, and D, we find $A = 1$, $B = -4$, $C = 7$, and $D = -1/4$. The particular solution is $y_p(x) = x^2 - 4x + 7 - e^x/4$. The general solution is then $y(x) = y_h(x) + y_p(x) \Rightarrow$ $y(x) = C_1 e^{-x} + C_2 x e^{-x} + x^2 - 4x + 7 - e^x/4$. Applying the initial conditions we find $C_1 = -27/4$, $C_2 = -1/2$, and the solution to the initial value problem: $y(x) = -27e^{-x}/4 - xe^{-x}/2 + x^2 - 4x + 7 - e^x/4$.

(e) First consider the homogeneous equation $y_h'' - 2y_h' + y_h = 0$. The associated characteristic equation is $r^2 - 2r + 1 = 0$ which has $r = 1$ as a repeated root. The homogeneous solution is then $y_h(x) = C_1e^x + C_2xe^x$. To find the particular solution to the inhomogeneous equation $y_p'' + 2y_p' + y_p = 8e^x$ pose the guess $y_p(x) = Ax^2e^x$. Substituting the guess into the equation, noting that $y_p'(x) = Ax^2e^x + 2Axe^x$ and $y_p''(x) = Ax^2e^x + 4Axe^x + 2Ae^x$, we obtain $(Ax^2e^x + 4Axe^x + 2Ae^x) - 2(Ax^2e^x + 2Axe^x) + (Ax^2e^x) = 8e^x$ $\Rightarrow 2Ae^x = 8e^x$. So $2A = 8$ or $A = 4$ and the particular solution is $y_p(x) = 4x^2e^x$. The general solution is then $y(x) = y_h(x) + y_p(x) \Rightarrow y(x) = C_1e^x + C_2xe^x + 4x^2e^x$. Applying the initial conditions we find $C_1 = 3$, $C_2 = -1$, and the solution to the initial value problem: $y(x) = 3e^x - xe^x + 4x^2e^x$.

(f) First consider the homogeneous equation $y_h'' + 2y_h' + 2y_h = 0$. The associated characteristic equation is $r^2 + 2r + 2 = 0$ which has which has roots $r_1 = -1 + i$, $r_2 = -1 - i$, and $r_3 = -3$. The homogeneous solution is then $y_h(x) = C_1e^{-x}\cos(x) + C_2e^{-x}\sin(x)$. To find the particular solution to the inhomogeneous equation $y_p'' + 2y_p' + 2y_p = 5\cos(2x)$ pose the guess $y_p(x) = A\cos(2x) + B\sin(2x)$. Substituting the guess into the equation, noting that $y_p'(x) = -2A\sin(2x) + 2B\cos(2x)$ and $y_p''(x) = -4A\cos(2x) - 4B\sin(2x)$, we obtain $(-4A\cos(2x) - 4B\sin(2x)) + 2(-2A\sin(2x) + 2B\cos(2x)) + 2(A\cos(2x) + B\sin(2x)) = 5\cos(2x) \Rightarrow$ $(-2A + 4B)\cos(2x) + (-4A - 2B)\sin(2x) = 5\cos(2x)$. So $-2A + 4B = 5$, $-4A - 2B = 0$, and, solving for A and B we find $A = -1/2$, $B = 1$. The particular solution is $y_p(x) = -\cos(2x)/2 + \sin(2x)/2$. The general solution is then $y(x) = y_h(x) + y_p(x) \Rightarrow y(x) = C_1e^{-x}\cos(x) + C_2e^{-x}\sin(x) - \cos(2x)/2 + \sin(2x)/2$. Applying the initial conditions we find $C_1 = 0$, $C_2 = e^\pi$, and the solution to the initial value problem: $y(x) = e^{-x+\pi}\sin(x) - \cos(2x)/2 + \sin(2x)/2$.

Problem 2: The suspension in a car can be modeled as a vibrating spring with damping due to the shock absorbers. This leads to the equation for the vertical displacement $x(t)$ at time t,

$$mx''(t) + bx'(t) + kx(t) = 0,$$

where m is the mass of the car, b is the damping constant of the shocks, and k is the spring constant. If the mass m of the car is 1000kg and the spring constant k is 3000kg/s^2, determine the minimum value for the damping constant b in kilograms per seconds that will provide a smooth, *oscillation-free* ride. If we replace the springs with heavy-duty ones having twice the spring constant k, how will this minumum change?

Problem 2 Solution:
The equation for x is $mx'' + bx' + kx = 0$. You could plug in the numerical values for the mass m and spring constant k and solve, that would be fine. However here we leave the in the symbols so that we can answer both questions at once.
The characteristic equation is $mr^2 + br + k = 0$ which has roots

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

For there to be no oscillations we need $r_{1,2}$ to be real, so we require that $b^2 - 4mk \geq 0$. Therefore the minimum value for the damping constant b that will provide a smooth, *oscillation-free* ride (also known as the "critical damping") is given by $b_{min}^2 - 4mk = 0$. Thus we find that minimum $b_{min} = 2\sqrt{mk}$. With the values for mass and spring constant k given we obtain the minumum damping $b_{min} = 1000\sqrt{3}$kg/s. If the spring constant doubles i.e. the spring constant is now $\tilde{k} = 2k$, then the minimum damping is $b_{min} = 2\sqrt{m\tilde{k}} = 2\sqrt{2mk}$. So if the springs are replaced with heavy-duty ones with double the spring constant, the minimum damping for no oscillations INCREASES by a factor of $\sqrt{2}$. The value for that larger minumum damping, given a mass $m = 1000kg$ and a spring constant $\tilde{k} = 2k = 6000\text{kg/s}^2$, is $b_{min} = 1000\sqrt{6}\text{kg/s}$.

Problem 3: A vibrating spring *without* damping can be modeled by the initial value problem:

$$my''(t) + ky(t) = 0 \quad y(0) = y_0, \quad y'(0) = y_1$$

for m the mass of the spring and k is the spring constant.
(a) If $m = 10\text{kg}$, $k = 250\text{kg/s}^2$, $y_0 = 0.3\text{m}$, and $y_1 = -0.1\text{m/s}$, find the equation of motion $y(t)$ for this undamped vibrating spring.
(b) What is the frequency of oscillation of this spring system?

Problem 3 Solution:
(a) As discussed in class, the characteristic equation for $my''(t) + ky(t) = 0$ is $mr^2 + k = 0$ which has roots $r = \pm i\sqrt{k/m}$ (both the spring constant k and the mass m are positive). The solution is therefore $y(t) = C_1\cos(\omega t) + C_2\sin(\omega t)$, where $\omega = \sqrt{k/m}$ is the frequency of oscillation. As we are given $k = 250\text{kg/s}^2$ and $m = 10\text{kg}$, $\omega = \sqrt{(250\text{kg/s}^2)/(10\text{kg})} = 5\text{s}^{-1}$. Applying the initial conditions $y(0) = 0.3\text{m}$ and $y(0) = -0.1\text{m/s}$ we find $C_1 = 0.3$ and $C_2 = -0.02$. Thus the equation of motion for this undamped vibrating spring is $y(t) = 0.3\cos(5t) - 0.02\sin(5t)$. Alternatively we could right the solution as $y(t) = R\cos(5t + \delta)$. Applying the initial conditions we find $R = \sqrt{0.3^2 + 0.02^2}$ and $\delta = \tan^{-1}(-0.02/.3) = -\tan^{-1}(1/150)$; the solution is $y(t) = \sqrt{0.3^2 + 0.02^2}\cos(5t - \tan^{-1}(1/150))$. Either way of expressing $y(t)$ is fine.

(b) As mentioned in the solution of part (a), the frequency of oscillation is $\omega = 5\text{s}^{-1}$.

Problem 4: A vibrating spring *with* damping can be modeled by the initial value problem:

$$my''(t) + by'(t) + ky(t) = 0 \quad y(0) = y_0, \quad y'(0) = y_1$$

for m the mass of the spring, k is the spring constant, and b the damping constant.
(a) Using the same values for m, k, y_0, and y_1 as in Problem 3, now with $b = 60\text{kg/s}$, find the equation of motion $y(t)$ for this damped vibrating spring.
(b) What is the frequency of oscillation of this spring system?
(c) Compare the results of problems 3 and 4 and determine what effect the damping has on the frequency of oscillation. What other effects does it have on the solution? What is the long-time behaviour of the solution (behaviour of the solution as $t \to \infty$)?

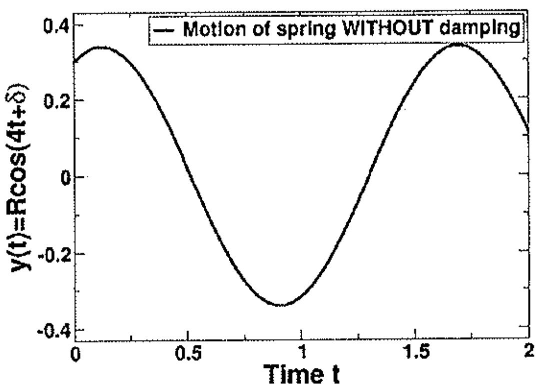
Problem 4 Solution:
(a) As in Problem 2 the characteristic equation is $mr^2 + br + k = 0$ which has roots

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}.$$

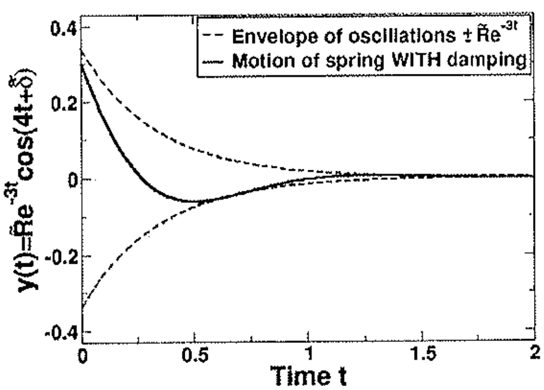
We are given that $m = 10$kg, $k = 250$kg/s, and $b = 60$kg/s so the roots are $r_{1,2} = -3 \pm 4i$. Note that $b^2 - 4mk = -6400 < 0$: The damping is small enough so that there are oscillations. The general solution to the differential equation is then $y(t) = C_1 e^{-3t}\cos(4t) + C_1 e^{-3t}\sin(4t)$. Applying the initial conditions $y(0) = 0.3$m and $y(0) = -0.1$m/s we find $C_1 = 0.3$ and $C_2 = 0.16$. Thus the equation of motion for this *damped* vibrating spring is $y(t) = 0.3e^{-3t}\cos(4t) + 0.16e^{-3t}\sin(4t)$. Alternatively we could right the solution as $y(t) = \tilde{R}e^{-3t}\cos(4t + \tilde{\delta})$. Applying the initial conditions we find $\tilde{R} = \sqrt{.3^2 + .16^2}$ and $\tilde{\delta} = \tan^{-1}(.16/.3) = \tan^{-1}(8/15)$; the equation of motion for this *damped* vibrating spring is $y(t) = \sqrt{.3^2 + .16^2}e^{-3t}\cos(4t + \tan^{-1}(8/15))$. Either way of expressing $y(t)$ is fine.

(b) The frequency of oscillation, or rather the "quasi-frequency," is $\sqrt{b^2 - 4mk}/2m = 4\text{s}^{-1}$.

(c) Note that small damping decreases the frequency of oscillation. The frequency of oscillation on the undamped vibrating spring is 5s^{-1} (from Problem 3), and the quasi-frequency of oscillation on the *damped* vibrating spring is 4s^{-1}.
Other effects of the damping:
(1) The period of the vibrating spring without damping is less than the quasi-period of the vibrating spring with damping. We see that in Problems 3 and this problem, $2\pi/5 < 2\pi/4$.
(2) Importantly, the damping *damps* the oscillations: the amplitude Re^{-3t} gets smaller and smaller as time goes on. See the plots:



Solution from Problem 3



Solution from Problem 4

Finally, the long-time behaviour of the solution for the motion of a *damped* spring, i.e. the behaviour of the solution as $t \to \infty$, is that the amplitude of the oscillations decreases with time and goes to zero. This actually happens quite rapidly - see the plot above.

<u>Problem 5</u> Consider the *nonhomogeneous* second order ODE

$$y'' - 2y' - 3y = 2e^{-t}.$$

The general solution of this equation is $y(t) = c_1y_1(t) + c_2y_2(t) + Y_p(t)$ where $y_1(t), y_2(t)$ are a fundamental set of solutions to the corresponding homogeneous ODE and $Y_p(t)$ is a particular solution to the nonhomogeneous ODE.

(a) Suppose we "guess" a form for the particular solution as $Y_p(t) = Ae^{-t}$ (since this is similar to the form of the time-dependent forcing term.) Plug this function into the ODE and show that you arrive at a contradiction. Why does this happen?

(b) Now revise your guess to the form $Y_p(t) = Ate^{-t}$. Show that this works, find the value of A, and thereby also find the general solution to the nonhomogeneous ODE.

Problem 6 Solution:

(a) Note that the corresponding homogeneous problem is $y'' - 2y' - 3y = 0$, which has characteristic equation $r^2 - 2r - 3 = (r-3)(r+1) = 0$ so $r = 3, -1$ are the roots, and the fundamental set of solutions is e^{-t}, e^{3t}. If we use $Y_p(t) = Ae^{-t}$, we get $Y_p'(t) = -Ae^{-t}, Y_p''(t) = Ae^{-t}$. plugging into the ODE leads to $Ae^{-t} - 2(-Ae^{-t}) - 3(Ae^{-t}) = 2e^{-t}$. Canceling a factor of e^{-t} and simplifying leads to $0 = 2e^{-t}$ which is a contradiction. This stems from the fact that $y_1(t) = e^{-t}$ is a solution to the homogeneous problem.

(b) Now assume that $Y_p(t) = Ate^{-t}$. Then the derivatives we need are $Y_p'(t) = A(1-t)e^{-t}$ and $Y_p''(t) = Ae^{-t}(t-2)$. Sub these into the nonhomogeneous ODE to get (after canceling the exponential factor): $(At-2A)-2(A-At)-3At = 2$. This has to hold for all t. Rewrite it as $(A+2A-3A)t-4A = 2$. Note that the coefficient of t is zero, so equation simplifies to $-4A = 2$ so $A = -1/2$ and $Y_p(t) = -(1/2)te^{-t}$. The general solution is thus

$y(t) = c_1y_1(t) + c_2y_2(t) + Y_p(t) = c_1e^{-t} + c_2e^{3t} - (1/2)te^{-t}$

Problem 6

Consider a circuit with a resistor, inductor, and capacitor in series (Fig ??), and suppose this is connected to a constant voltage V. Recall that the ODE satisfied by the charge $q(t)$ across the capacitor in such a circuit is

$$Lq'' + Rq' + (1/C)q = V.$$

Also recall that the current $I(t)$ in the circuit is related to the charge $q(t)$ by $I = q'(t)$.

(a) Use the above information to find the differential equation satisfied by the current $I(t)$.

(b) Consider the (unrealistic) case that the resistance in $R = 0$ in this circuit. Determine the behaviour of $I(t)$, i.e. find the general solution to the equation you found in part (a). What is the frequency of the oscillation?

(c) Now suppose that R is gradually increased. At what value of R will there be *no oscillation*? Sketch the behaviour of $I(t)$ for values of R below and above that critical value.

(d) Someone has set up the circuit with ($R \neq 0$) so that there is initially a charge on the capacitor when a switch is closed so that at time $t = 0$, $q(0) = q_0$ and $I(0) = I_0$ are known. Find $I(t)$ using these initial conditions.

Problem 6 Solution:

(a) Differentiate both sides with respect to t and use $I = q'(t)$ to obtain $LI'' + RI' + (1/C)I = 0$ (since V is constant).

(b) The characteristic equation is $Lr^2+Rr+(1/C)=0$ with roots $r_{1,2}=\dfrac{-R\pm\sqrt{R^2-4(L/C)}}{2L}$. In the case of $R=0$ these roots are $r_{1,2}=\pm\sqrt{1/(LC)}i$ which are pure imaginary. Then defining $\omega=\sqrt{1/(LC)}$ we have a general solution $I(t)=c_1\cos(\omega t)+c_2\sin(\omega t)$.

(c) The oscillations will cease when the roots are no longer complex, i.e. when $R^2-4(L/C)=0$ so when $R=2\sqrt{L/C}$.

(d) If $R\neq 0$ then the general solution is $I(t)=e^{\sigma t}[c_1\cos(\omega t)+c_2\sin(\omega t)]$ where

$$\sigma=-R/2L,\quad \omega=\frac{\sqrt{|R^2-4(L/C)|}}{2L}$$

We have two initial conditions, but one of them is in terms of the charge. The latter can not be used directly as the ODE is for the current $I(t)$. However, we can use the alternate equation $Lq''+Rq'+(1/C)q=V$ together with $I(t)=q'(t)$ to note that $LI'+RI+(1/C)q=V$ and in particular, at time $t=0$, this means that $LI'(0)+RI(0)+(1/C)q(0)=V(0)=V$ (since V is constant). Thus, we can actually rewrite one of the initial conditions as $I'(0)=\frac{V-(RI_0+(1/C)q_0)}{L}\equiv A$. (We define A to stand for this combination of constants.) We now find the constants c_1,c_2 using initial conditions. This leads to the system of equations $c_1=I_0, \sigma c_1+\omega c_2=A$. We find that $c_2=\frac{-I_0\sigma+A}{\omega}$ and obtain the desired solution, $I(t)=e^{\sigma t}[I_0\cos(\omega t)+\frac{A-I_0\sigma}{\omega}\sin(\omega t)]$.

Problem 7 Consider the circuit shown in Fig ?? and assume that $V=0$ and a switch is closed at $t=0$. In this circuit, the inductance is $L=0.05$ Henrys, Capacitance is $C=2\times10^{-4}$ Farads and the resistance is $R=10\Omega$. The initial charge on the capacitor at time $t=0$ is 2 coulombs. Determine the current $I(t)$ for $t\geq 0$.

Problem 7 Solution: We write the differential equation for $q(t)$ as follows: $Lq''+Rq'+q/C=V=0$ We have $R^2-4L/C=10^2-4\cdot(0.05)/2\cdot10^{-4}=-900$. This means that the circuit is underdamped and $\omega=Im(r)=\sqrt{900}/(2\cdot0.05)=300$, $\sigma=Re(r)=-R/2L=-10/(2\cdot0.05)=-100$. Thus

$$q(t)=e^{-100t}(c_1\cos(300t)+c_2\sin(300t))$$

Until the switch is closed, we have $I(0)=q'(0)=0$ and $q(0)=2$. Thus $2=q(0)=c_1, 0=q'(0)=300c_2-100c_1$, so $c_1=2, c_2=2/3$ and the solution is

$$q(t)=e^{-100t}(2\cos(300t)+(2/3)\sin(300t)).$$

Now we can find that the current, by differentiating the above, to arrive at

$$i(t)=q'(t)=-100e^{-100t}(2\cos(300t)+(2/3)\sin(300t))+e^{-100t}(-600\sin(300t)+200\cos(300t))=-6666e^{-100t}\sin(300t)$$