HOMEWORK 2: MATH 265, L Keshet (**Final version**) Due in class on September 29, 2010

NOTE: Most problems on this assignment are straightforward. Problem 4 may take a bit more time and effort.

Problem 1: In each case, solve the following second order ODEs for $y(t)$:

(a) $y'' + 2y' - 3y = 0$, and $y(0) = 1, y'(0) = 2$

(b) $y'' - 9y' + 20y = 0$ and $y(0) = 1, y'(0) = 0$

(c) $y'' - 2y' + 5y = 0$ and $y(0) = 1, y'(0) = 1$.

(d) $y'' - 2y = 0$ and $y(0) = 0, y'(0) = 2$.

Solution to Problem 1:

(a) The characteristic equation is $r^2 + 2r - 3 = 0$. This factors into $(r + 3)(r - 1) = 0$ so has solutions $r = 1, -3$. The general solution is thus $y(t) = C_1 e^t + C_2 e^{-3t}$. We find C_1, C_2 from the initial conditions. We need to find $y'(t)$ by differentiating $y(t)$: we get $y'(t) = C_1 e^t - 3C_2 e^{-3t}$. Using the initial conditions, we have $y(0) = 1, \Rightarrow 1 = C_1 + C_2$ and $y'(0) = 2 \Rightarrow 2 = C_1 - 3C_2$. Solving these equations leads to $C_2 = -1/4$ and $C_1 = 5/4$ so the solution is $y(t) = \frac{5}{4} e^t + \frac{1}{4} e^{-3t}$

(b) The characteristic equation is $r^2 - 9r + 20 = (r - 4)(r - 5) = 0$, so the roots are $r = 4, 5$ and the general solution is $y(t) = C_1 e^{4t} + C_2 e^{5t}$. Then the initial conditions mean that $y(0) = 1 = C_1 + C_2$ and $y'(0) = 0 = 4C_1 + 5C_2$. Solving these for C_1, C_2 leads to $y(t) = 5e^{4t} - 4e^{5t}$.

(c) The characteristic equation is $r^2 - 2y + 5 = 0$. This has the complex roots $r = 1 \pm 2i$ so the general solution is $y(t) = C_1 e^t \sin(2t) + C_2 e^t \cos(2t)$. We also need the derivative $y'(t) = e^t (C_1 \sin(2t) + 2C_1 \cos(2t) + C_2 \cos(2t) - 2C_2 \sin(2t))$. Then we use the initial conditions: $1 = y(0) = C_2$ and $1 = (2C_1 + C_2)$ (we used the facts that $e^0 = 1, \sin(0) = 0, \cos(0) = 1$). This tells us that $C_2 = 1, C_1 = 0$ so the solution is $y(t) = e^t \cos(2t)$.

(d) Characteristic equation: $r^2 - 2 = 0$ so $r = \pm\sqrt{2}$ and $y(t) = C_1 e^{\sqrt{2}t} + C_2 e^{-\sqrt{2}t}$. The derivative: $y'(t) = \sqrt{2} C_1 e^{\sqrt{2}t} - \sqrt{2} C_2 e^{-\sqrt{2}t}$. Using the I.C's: $y(0) = 0 = C_1 + C_2$ and $y'(0) = 2 = \sqrt{2} C_1 - \sqrt{2} C_2$. Solving for constants leads to $y(t) = \frac{\sqrt{2}}{2} e^{\sqrt{2}t} - \frac{\sqrt{2}}{2} e^{-\sqrt{2}t}$.

Problem 2: Consider the differential equation $ay'' + by' + cy = 0$. Suppose that the two functions $y = f_1(t)$ and $y = \frac{1}{2}[f_2(t) + f_1(t)]$ are both solutions to this equation. Show that the function $f_2(t)$ is also a solution.

Solution to Problem 2: We could just use the superposition principle, but the point of the question is to establish this from first principles.

Since one of the solutions is $\frac{1}{2}[f_2(t) + f_1(t)]$, it must be true that this function satisfies the ODE, i.e. when we compute its derivatives and plug it into the left hand side, we should get zero. Thus,

$$a \frac{d^2}{dt^2} \frac{1}{2}[f_2(t) + f_1(t)] + b \frac{d}{dt} \frac{1}{2}[f_2(t) + f_1(t)] + c \frac{1}{2}[f_2(t) + f_1(t)] = 0.$$

Let us rewrite this as $\frac{1}{2}\left[(af_2'' + bf_2' + cf_2) + (af_1'' + bf_1' + cf_1)\right] = 0$ where we have simply rearranged terms. But we are told that f_1 is a solution so it must satisfy $(af_1'' + bf_1' + cf_1) = 0$. Subtracting this from the last equation leaves us with it must be true that $\frac{1}{2}\left[(af_2'' + bf_2' + cf_2)\right] = 0$ so $af_2'' + bf_2' + cf_2 = 0$ so we see that f_2 is also a solution.

Problem 3: Find a value of the constant r such that both e^{rt} and te^{rt} are solutions to the ODE

$$ay'' + by' + cy = 0$$

Solution to Problem 3: We already know that for e^{rt} to be a solution, r must satisfy the quadratic equation $ar^2 + br + c = 0$, i.e. $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Now for the other function, $y_2(t) = te^{rt}$ to be a solution, it too has to satisfy $ay'' + by' + cy = 0$ bf for all values of t. Computing the derivatives, we have $y_2'(t) = e^{rt}(1+rt)$ and $y_2''(t) = e^{rt}(2r+tr^2)$. Now substitute these into the ODE to make sure we satisfy that ODE: $ae^{rt}(2r+tr^2) + be^{rt}(1+rt) + cte^{rt} = 0$. (This has to be true for all t, and is exactly what it means to say that "te^{rt} is a solution".)

Now let us simplify and see what this says. We cancel out a factor of e^{rt} and rearrange to get
$a(2r+tr^2) + b(1+rt) + ct = (ar^2+br+c)t + (2ar+b) = 0$.

The only way this can be true **for all** t is if the coefficient in front of t is zero, and the constant term is zero.

We already know that the first term is zero (by the argument above) so it must be also true that $2ar+b = 0$, i.e. $r = -b/a$. Note that this happens exactly when the quadratic equation has equal roots, i.e. when $b^2 = 4ac$.

Problem 4: A patient is in the hospital on intravenous medication. We will denote by $I(t)$ the rate at which medication is infused (injected into the patient). Assume this is already corrected for weight of patient and that it is immediately well-mixed in the bloodstream. Let $c(t)$ denote the drug concentration (mg/L) in the bloodstream at time t. The drug is broken down by the liver at a constant rate $r \geq 0$ (per hr). Assume that the ODE and initial condition that describes this situation is

$$\frac{dc}{dt} = I(t) - rc, \quad c(0) = 0.$$

(a) Suppose that $I(t)$ is switched on at time 0, is constant for an hour ($I(t) = \bar{I}$ for $0 \leq t \leq 1$hr) and then switched off. Find the value of $c(t)$ during and after this period of time. (Your answer will be in terms of constants in the problem.

(b) Sketch the solution you got in part (a). (The sketch should be approximate but should be labeled carefully.)

(c) Now suppose that for a second patient, the infusion rate is periodic and the decay rate is $r = 1$ per hour, so that the ODE is

$$\frac{dc}{dt} = 1 + \sin(\pi t/6) - c, \quad c(0) = 0.$$

Solve the ODE for $c(t)$ sketch the solution.

Solution to Problem 4:

(a) For the first hour we have that $I(t) = \bar{I}$ so the problem is then $\frac{dc}{dt} = \bar{I} - rc, \quad c(0) = 0$. This can be solved by a number of methods, e.g. integrating factor, just like examples we have seen. The standard form of the equation is $\frac{dc}{dt} + rc = \bar{I}$, so the integrating factor is $\mu(t) = e^{rt}$. The equation is then $\frac{d}{dt}[e^{rt}c(t)] = \bar{I}e^{rt}$, and integrating and other steps lead to $c(t) = \frac{\bar{I}}{r}(1 - e^{-rt})$. This holds for $0 \leq t \leq 1$. At $t = 1$ the value of c is $c(1) = \frac{\bar{I}}{r}(1 - e^{-r}) \equiv c_1$. (This is a constant that we have named c_1.) For $t \geq 1$ the infusion is zero, so the ODE and initial condition is then

$$\frac{dc}{dt} = -rc, \quad c(1) = c_1.$$

The solution is exponentially decreasing, $c(t) = Ke^{-rt}$ and we have that $c_1 = c(1) = Ke^{-r}$, so the constant can be found: $K = c_1 e^r = \frac{\bar{I}}{r}(1 - e^{-r})e^r = \frac{\bar{I}}{r}(e^r - 1)$. So for $t \geq 1$, $c(t) = Ke^{-rt} = [\frac{\bar{I}}{r}(e^r - 1)]e^{-rt}$. This is a decaying function that approaches $c = 0$ as $t \to \infty$.

(b) See Fig 1.

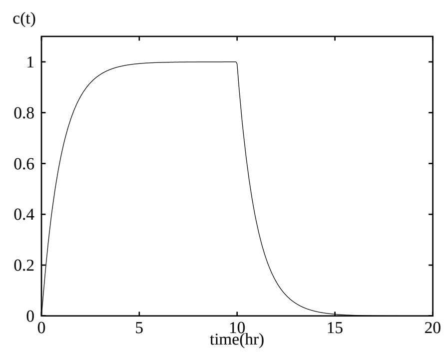

Figure 1: For problem 4 part (b). A sketch of $c(t)$ for $I(t)$ switched on at $t = 0$ and off at $t = 1$.

(c) We put the function in standard form $\frac{dc}{dt} + c = 1 + \sin(\pi t/6)$ and find as above that the integrating factor is $\mu(t) = e^t$. Similar steps lead to $\frac{d}{dt}[ce^t] = e^t(1 + \sin(\pi t/6))$. We must integrated both sides. This requires integration by parts for one term. **See below for a reminder how to do this**.

Then

$$[ce^t] = \int [e^t(1 + \sin(\pi t/6))]dt + K = e^t \left[1 + \frac{-6\pi\cos(\pi t/6) + 36\sin(6\pi t)}{36 + \pi^2}\right] + K$$

Thus $c(t) = \left[1 + \frac{-6\pi\cos(\pi t/6) + 36\sin(6\pi t)}{36 + \pi^2}\right] + Ke^{-t}$.

At time $t = 0$ we have $0 = c(0) = \left[1 + \frac{-6\pi}{36 + \pi^2}\right] + K$.

(where we have used that $\sin(0) = 0, \cos(0) = 1, e^0 = 1$.) Thus $K = -\left[1 + \frac{-6\pi}{36+\pi^2}\right]$ and

$$c(t) = \left[1 + \frac{-6\pi\cos(\pi t/6) + 36\sin(6\pi t)}{36 + \pi^2}\right] - \left[1 + \frac{-6\pi}{36 + \pi^2}\right] e^{-t}$$

A sketch is shown in Fig 2.

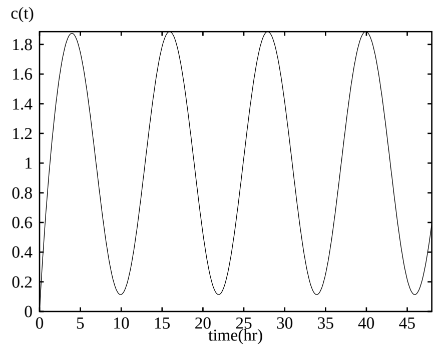

Figure 2: For problem 4 part (c). A sketch of $c(t)$ for $I(t) = 1 + \sin(\pi t/6)$ and $r = 1$

Integration by parts This is treated in any standard calculus book. Set $u = e^x, dv = \sin(x)dx$ You will find that you need two steps, each involving a similar integral with either the sine or cosine function. In the second step, you need $u = e^x, dv = \cos(x)dx$ Calling these integrals I_1, I_2 we have

$$I_1 \equiv \int e^x \sin(x)dx = -e^x \cos(x) + \int e^x \cos(x)dx, \quad \text{AND} \quad I_2 \equiv \int e^x \cos(x)dx = e^x \sin(x) - \int e^x \sin(x)dx$$

Thus $I_1 = -e^x \cos(x) + I_2$, while $I_2 = e^x \sin(x) - I_1$. Solve these for I_1, I_2 to get

$$I_1 \equiv \int e^x \sin(x)dx = \frac{1}{2}e^x(\sin(x) - \cos(x)) + C$$

and similarly for I_2.

Problem 5: A student solves a certain linear homogeneous differential equation of second order (e.g. $y'' + p(t)y' + q(t)y = 0$) and finds two solutions: $y_1(t) = 2e^t$ and $y_2(t) = e^{t-1}$. Now he would like to find the constants c_1 and c_2 such that the solution $y(t) = c_1y_1(t) + c_2y_2(t)$ also satisfies the initial conditions $y(0) = 1, y'(0) = 1$. The student encounters some difficulty. What is the difficulty, and why does it occur? (Trace the steps that the student might be making and help figure out why he/she runs into problems).
Solution to Problem 5: The two functions cannot form a fundamental set of solutions since they are actually both constant multiples of the same exponential function: $y_2(t) = e^{t-1} = \frac{1}{e}e^t = \frac{1}{2e}y_1(t) = ky_1(t)$ for a constant k. Another way to say the same thing is that the wronskian of these functions is zero: $W = y_1y_2' - y_2y_1' = (2e^t)(e^{t-1})' - (2e^t)'(e^{t-1}) = (2e^t)(\frac{e^t}{e})' - (2e^t)'(\frac{e^t}{e}) = \frac{2}{e}[(e^t)(e^t)' - (e^t)'(e^t)] = 0$. (Remember that e is just a constant.)

Problem 6: Unlike linear ODEs, for which we have results guaranteeing the existence and "good behaviour" of solutions, **nonlinear** ODEs can have all kinds of problems. Consider the simple (nonlinear) ODE

$$\frac{dy}{dt} = y^2, \quad y(0) = y_0.$$

Solve this ODE using separation of variables. Show that the solution can "blow up" (become undefined) at some finite time. For what value of y_0 will the solution "blow up" when $t = 2$?

Solution to Problem 6: The solutions by separation of variables is as follows:

$$\frac{dy}{dt} = y^2, \quad \Rightarrow \frac{dy}{y^2} = dt, \quad \Rightarrow \int \frac{1}{y^2} dy = \int y^{-2} dy = \int dt + C, \quad \Rightarrow -y^{-1} = t + C \quad \Rightarrow y(t) = -\frac{1}{t+C}.$$

Using the initial condition $y(0) = y_0$, we find that the constant is $C = -1/y_0$ so we arrive at the solution $y(t) = \frac{1}{(1/y_0)-t}$. There is a problem when the denominator is zero, which will happen when $t = (1/y_0)$. For example, if the initial condition is $y(0) = y_0 = 1/2$, then the solution "blows up" when $t = 2$.