HOMEWORK 1: MATH 265, L Keshet Due in class on September 22, 2010

Problem 1: In each case, solve for $y(t)$:

(a) $y' + 3y = 2e^{t/2}$ with $y(0) = 1$.

(b) $y' - 4y = t$ with $y(1) = 0$.

(c) $ty' + 2y = \sin t$ with $y(\pi/2) = 0$.

(d) $t^2 y' + 2ty = t^3 + 1$ with $y(1) = 1$.

Problem 2: Draw the direction fields for each of the below and sketch the solution curve corresponding to the indicated initial condition.

(a) $y' = y - 2\cos(t) + 1$, $y(0) = 0$

(b) $y' = \sqrt{y} - ty$, $y(0) = 2$

Problem 3: The following equation, called the **logistic equation**, is often used to describe the growth of a population $N(t) \geq 0$ with limited resources:

$$\frac{dN}{dt} = rN\frac{(K - N)}{K}.$$

Here $r, K > 0$ are constants (called the intrinsic growth rate and the carrying capacity, respectively).

(a) Define a new variable, $y(t) = N(t)/K$ and rewrite the logistic equation in terms of this new variable.

(b) Sketch a direction field for $t \geq 0, y \geq 0$ and include on it the solution curves for initial condition $y(0) = 0, y(0) = 0.2, y(0) = 0.8, y(0) = 1.6$. What happens as $t \to \infty$? Interpret in terms of the growing population (i.e. in terms of the original variable $N(t)$.)

(c) Find a mathematical argument that supports the following statement: "Only solutions curves with $y(0) \leq 1/2$ have an inflection point."

Problem 4: According to Torricelli's Law, the height of fluid in a container above a hole (through which the fluid is escaping) is governed by a differential equation:

$$\frac{dh}{dt} = -k\sqrt{h}.$$

where $k \geq 0$ is a constant. Suppose the height of the fluid is initially $h(0) = h_0$. How long does it take for the fluid to drain to the level of the hole?

Problem 5: Set up the following two problems as initial value problems (i.e. differential equation and initial condition) and solve each one.

(a) In the LR circuit shown in Fig 1(a), $R, L, V \geq 0$ are constant. Find the current $i(t)$ given that $i(0) = 0$.

(b) In the RC circuit shown in Fig 1(b), $R, C \geq 0$ are constant and the voltage is time dependent, $V(t) = e^{-t}$. Find the charge on the capacitor $q(t)$ given that $q(0) = 0$.

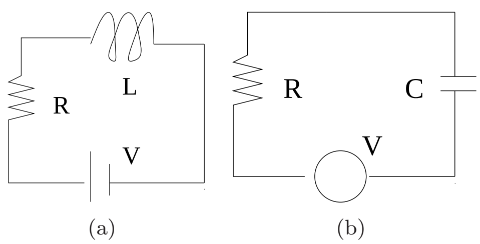

Figure 1: For problem 5 (a) and 5 (b)

Problem 6: Consider a stirred tank reactor that initially contains a volume $V(0) = V_0$ of water. Now suppose that a stock solution of salt (at concentration S gm/Litre) is pumped in at rate $F_{in} = F$ Litres/hr and the well-stirred mixture is pumped out at a slightly faster rate, $F_{out} = (F + f)$ Litres/hr where $f > 0$. [Note that the volume of the fluid will not be constant.] Let $C(t)$ denote the concentration of salt inside the tank.

(a) Set up this problem as a differential equation problem for the volume of fluid $V(t)$ and the salt concentration $C(t)$.

(b) Determine the volume of fluid $V(t)$ for $t > 0$. Over what period of time $0 \leq t \leq T$ is this result valid?

(c) Use your result in (b) to set up one ODE that depends only on the variable $C(t)$ and solve that equation.