Sept 27

We have been studying the initial value problem (IVP) consisting of 2nd order linear diffl' eqn

$$ay'' + by' + cy = 0 \qquad \text{with initial conds. (I.C's)} \quad \begin{cases} y(0) = y_0 \\ y'(0) = y_0' \end{cases}$$

Solns are of the form $y = e^{rt}$ $\quad ar^2 + br + c = 0 \leftarrow$ character eqn.

so $\qquad r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$y_1(t) = e^{r_1 t} \qquad y_2(t) = e^{r_2 t} \qquad \text{are two possible solns}$$

These solns form a fundamental set $\Leftrightarrow$ every soln can be expressed as $y(t) = c_1 y_1(t) + c_2 y_2(t)$ $\quad c_1, c_2$ constants

$\Leftrightarrow$ We can always solve for c_1, c_2 (uniquely) given I.C's

$$\Leftrightarrow \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

[Note: for $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$

$$W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = e^{r_1 t} e^{r_2 t} (r_2 - r_1) \neq 0 \quad \text{provided } r_1 \neq r_2$$
]

Conclusion: If the roots of char. eqn are different $(r_1 \neq r_2)$ then $e^{r_1 t}, e^{r_2 t}$ form a fundamental set of solns, and so the gen'l soln to the IVP is $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

We now go back to exploring possible cases for various types of roots.

$$ay'' + by' + cy = 0 \qquad ar^2 + br + c = 0 \quad \text{char eqn}$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Consider case
$\underline{b^2 - 4ac < 0}$

then $r = \sigma \pm i\omega$ ← pair of complex conj roots

where $\sigma = -\dfrac{b}{2a}$, $\omega = \dfrac{\sqrt{|b^2 - 4ac|}}{2a}$

so the two solns:

$$y_1(t) = e^{r_1 t} = e^{(\sigma + i\omega)t} = e^{\sigma t} e^{i\omega t}$$
$$y_2(t) \qquad = e^{(\sigma - i\omega)t} = e^{\sigma t} e^{-i\omega t}$$

$$\left.\begin{aligned} \tilde{y}_1(t) &= e^{\sigma t}(\cos(\omega t) + i\sin(\omega t)) \\ \tilde{y}_2(t) &= e^{\sigma t}(\cos(\omega t) - i\sin(\omega t)) \end{aligned}\right\} \text{ complex valued.}$$

but we can define a new pair.

$$y_1(t) = \frac{\tilde{y}_1(t) + \tilde{y}_2(t)}{2} = e^{\sigma t}\cos(\omega t)$$

$$y_2(t) = \frac{\tilde{y}_1(t) - \tilde{y}_2(t)}{2i} = e^{\sigma t}\sin(\omega t)$$

By Superp. Prin. these are also solns, Now real valued, (more convenient).

So gen'l soln to the ODE is of form

$$y(t) = e^{\sigma t}(c_1\cos(\omega t) + c_2\sin(\omega t))$$

Example: $y'' + 6y = 0$ $\quad$ I.C. → $y(0) = 2,\ y'(0) = 1$ $\quad$ (IVP)

char eq: $r^2 + 6 = 0$ $\quad r^2 = -6$ $\quad r = \pm i\sqrt{6}$

$\sigma = \mathrm{Re}(r) = 0$
$\omega = \mathrm{Im}(r) = \sqrt{6}$

$$y(t) = c_1\cos(\sqrt{6}t) + c_2\sin(\sqrt{6}t)$$

want to find c_1, c_2 from I.C.

Need $y'(t) = -c_1\sqrt{6}\sin(\sqrt{6}t) + c_2\sqrt{6}\cos(\sqrt{6}t)$

Sept 27

$2 = y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1$ $\qquad c_1 = 2$

$1 = y'(0) = \cancel{-c_1\sqrt{6}\sin(0)} + c_2\sqrt{6}\cos(0) = c_2\sqrt{6}$ $\qquad c_2 = \frac{1}{\sqrt{6}}$

so $y(t) = 2\cos(\sqrt{6}\,t) + \frac{1}{\sqrt{6}}\sin(\sqrt{6}\,t)$

Example 2: Add "damping"

$$y'' + \boxed{2y'} + 6y = 0 \qquad y(0) = 2 \quad y'(0) = 1$$

$$r^2 + 2r + 6 = 0 \implies r = \underbrace{-1}_{\sigma} \pm i\underbrace{\sqrt{5}}_{\omega}$$

"quasi-frequency"

$$\boxed{y(t) = e^{\sigma t}(c_1\cos(\omega t) + c_2\sin(\omega t))}$$

solve for c_1, c_2 $\qquad c_1 = 2 \qquad c_2 = \frac{3\sqrt{5}}{5}$

$$y(t) = \cancel{2}\, e^{-t}\left(2\cos(\sqrt{5}\,t) + \frac{3\sqrt{5}}{5}\sin\sqrt{5}\,t\right)$$

decreasing oscillations.

More generally.

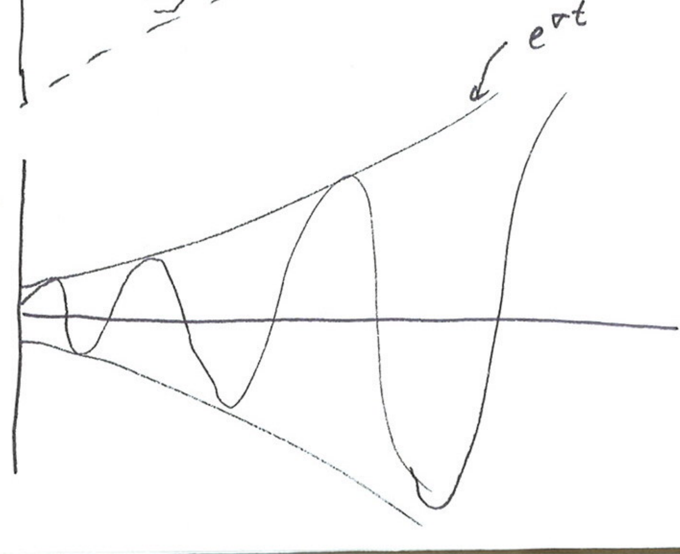
$y(t) = e^{\sigma t}(c_1\cos\omega t + c_2\sin\omega t)$

$y(0) = y_0$

$y'(0) = y_0' =$ slope of /

$\sigma < 0$

$\sigma > 0$



Application: spring-mass sys.

$my'' + cy' + ky = 0$ $\quad\quad y(0) = y_0 \quad y'(0) = y_0'$

$m, c, k \geq 0$ $\quad\quad m$ = mass, c = damping, k = spr. const.

char. eq. $\quad mr^2 + cr + k = 0$

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

σ

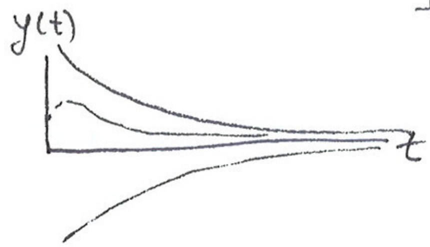
Cases:

(1) $c^2 - 4mk > 0$ $\quad\quad r_1, r_2$ real, negative. (distinct)

overdamped

decaying exp.

soln:



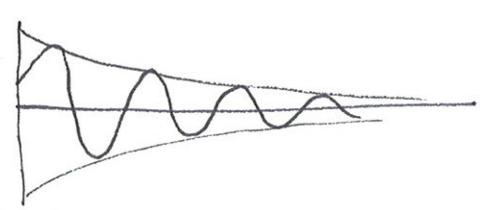
(2) $c^2 - 4mk = 0$ $\quad\quad r = -\frac{c}{2m}$ $\quad$ repeated roots $\quad \to$ next time.

critically damped

(3) $c^2 - 4mk < 0$ $\quad\quad$ solns $\quad y(t) = e^{\sigma t}(c_1 \cos(\omega t) + c_2 \sin(\omega t))$

$\sigma < 0$

underdamped



HW: it all applies to LRC circuit.