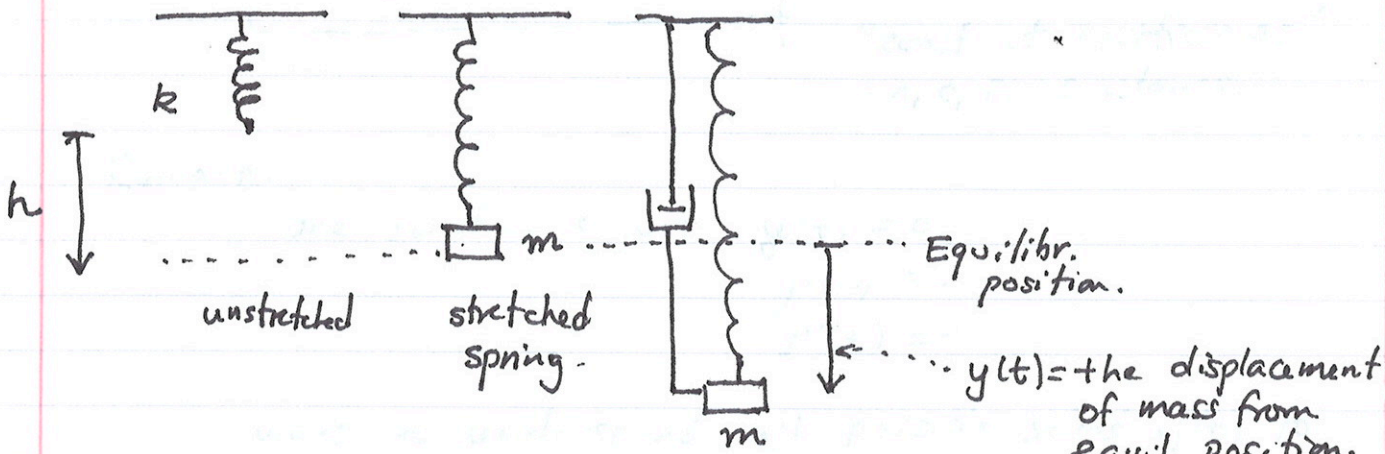
Sept 20, 2010 Second Order linear ODE's

Mass on a spring.



We'll take y positive in <u>downwards</u> direction.

$\therefore$ veloc positive "

accel. " "

<u>Assume:</u>

- mass m (kg) moves vertically
- $y(t)$ (meters) is vertical displ. from equil.
- damping force proportional to veloc. and impedes veloc.

$$F_{damp} = -cv$$

- Spring force $F_{spring} = -ky$ (opposite to displacem).
- forces : units of Newtons.
- mass of spring negligible.

Derive an ODE for $y(t)$

Use:

Newton's 2nd law Net Force = mass $\cdot$ accel.

define veloc $v(t) = \dfrac{dy}{dt}$

accel $a(t) = \dfrac{d^2y}{dt^2}$

$y(t)$ and its derivatives:

$$ma + \cancel{k}cv + ky = 0$$

$$\boxed{m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0}$$

describes the displ. $y(t)$

ODE : 2nd order. <u>Linear</u>

"constant coefficients"

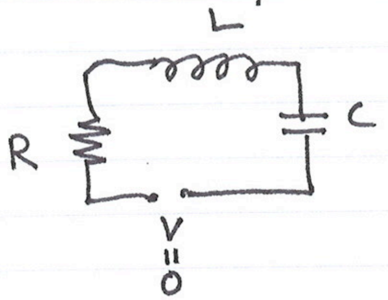
m, c, k constants.

Remark:

one solution is just
$$y(t) = 0$$
$$y'(t) = 0$$
$$y''(t) = 0$$

want to understand <u>all</u> possible solns $y(t)$ to the above ODE.

Another example of 2nd order linear ODE from Sept 8:



$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$ no applied voltage.

$L, R, C \geq 0$ constants

$q(t)$ – charge on capacitor

Remark: simple solution: $q(t) = 0$, $i(t) = 0$

we want to find all other solns.

Let us discuss ~~some~~ more generally:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

2nd order linear ODE

"homogeneous" eqn

a, b, c constants

(*) $ay'' + by' + cy = 0$ no explicit $f(t)$ in equation.

Solving? Try solns of form $y = e^{rt}$

a type of soln to 1st order lin eq. $\frac{dy}{dt} = ry$

$y(t) = e^{rt}$

$y'(t) = re^{rt}$ plug into (*)

$y''(t) = r^2e^{rt}$

$$ar^2\cancel{e^{rt}} + br\cancel{e^{rt}} + c\cancel{e^{rt}} = 0$$

recall $e^{rt} \neq 0$ so can cancel it $\Rightarrow$

characteristic eqn. $\boxed{ar^2 + br + c = 0}$ quadr. eqn in r

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ two possible values

i.e. both $f_1(t) = e^{r_1 t}$ and $f_2(t) = e^{r_2 t}$ would satisfy ODE.

Lin ODE $\Rightarrow$ <u>Superposition Principle</u>

for a <u>Linear</u> ODE like ⊛ if $f_1(t)$ and $f_2(t)$ are both solns. then also

$$y(t) = \underbrace{c_1 f_1(t) + c_2 f_2(t)}, \quad c_1, c_2 \text{ constants}$$

is also a soln.

"Linear superposition of $\underbrace{f_1(t),\ f_2(t)}$"

<u>Proof</u>: see book.
HW 2

(fine print $\to$) $\leftarrow$ "A fundamental set of solns"

<u>Example</u>: Solve $2y'' + y' - y = 0$ (Find $y(t)$)

assume $y(t) = e^{rt}$ plug in

$$2r^2\cancel{e^{rt}} + r\cancel{e^{rt}} - \cancel{e^{rt}} = 0$$

char. eqn $\quad 2r^2 + r - 1 = 0$

$$r = \frac{-1 \pm \sqrt{1 + 4.2}}{2.2} = -1, \tfrac{1}{2}$$

Solns: $\quad e^{-t}, \quad e^{\frac{1}{2}t}$

All solns can be expressed as superpositn.

$$y(t) = c_1 e^{-t} + c_2 e^{\frac{1}{2}t}$$

"general soln"

need <u>two</u> I.C.'s to find c_1, c_2

e.g. $y(0) = 3 \quad y'(0) = 7$

Find c_1, c_2.

Using Initial Conditions to solve 2nd order, Lin. ODE

Example for Sept 20

Solve $\underbrace{2y'' + y' - y = 0}_{\text{2nd order linear ODE}}$ $\qquad \underbrace{y(0) = 3 \quad y'(0) = -1}_{\text{initial conditions (I.C.'s)}}$

Look for solutions of form $y(t) = e^{rt}$. Plug y and its derivatives into the ODE, cancel common factor of e^{rt} to get

$$2r^2 + r - 1 = 0 \qquad \text{characteristic eqn.}$$

$$r = \frac{-1 \pm \sqrt{1 + 4\cdot 2}}{2\cdot 2} = \frac{-1 \pm \sqrt{9}}{4} = \frac{-1 \pm 3}{4} = -1, \tfrac{1}{2}$$

Solutions :

$$f_1(t) = e^{-t}, \quad f_2(t) = e^{\frac{1}{2}t}$$

so $\boxed{y(t) = C_1 e^{-t} + C_2 e^{\frac{1}{2}t}}$ ← the general solution (includes two arbitrary constants, C_1, C_2)

Now use I.C's to find C_1, C_2

$y(0) = 3 \quad \Rightarrow \quad 3 = C_1 e^0 + C_2 e^0 = C_1 + C_2 \qquad (1)$

$y'(0) = 1 \quad \Rightarrow \quad$ note $y'(t) = -C_1 e^{-t} + \frac{1}{2} C_2 e^{\frac{1}{2}t}$

$\Rightarrow \quad 1 = y'(0) = -C_1 \cdot e^0 + \frac{1}{2} C_2 e^0$

$= -C_1 + \frac{1}{2} C_2 \qquad (2)$

We have two algebraic eqns for the two constants:

$$\left.\begin{array}{ll} (1) & C_1 + C_2 = 3 \\ (2) & -C_1 + \frac{1}{2} C_2 = -1 \end{array}\right\} \text{solve for } C_1, C_2$$

(1)+(2) : $\frac{3}{2} C_2 = 2 \quad \Rightarrow \quad C_2 = \frac{4}{3}$

(1) : $C_1 = 3 - C_2 = 3 - \frac{4}{3} = \frac{5}{3}$

So the solution we want is $\boxed{y(t) = \frac{5}{3} e^{-t} + \frac{4}{3} e^{\frac{1}{2}t}}$