Sept 15

- More applications.
- Separable ODE's

Can be integrated by separating the dependent, indep. variables

Example 1

$$\frac{dy}{dt} = ry(1-y) \qquad \text{logistic eqn.}$$

$$\frac{dy}{y(1-y)} = r\,dt \qquad \text{separation}$$

Simply integrate both sides. (Partial fractions)

See HW 1 – dir. fields.

Example 2: $\frac{dy}{dt} = (1+y^2)(1+t^2)$ $\qquad y(0)=1 \leftarrow$ I.C.

$$\int \frac{dy}{1+y^2} = \int (1+t^2)\,dt + C$$

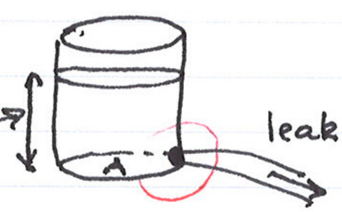
$$\arctan(y) = t + \frac{t^3}{3} + C$$

$$y = \tan\left(t + \frac{t^3}{3} + C\right) \qquad \text{I.C.} \Rightarrow C = \frac{\pi}{4}$$

Application: Deriving Torricelli's Law.

How long until fluid drains out.

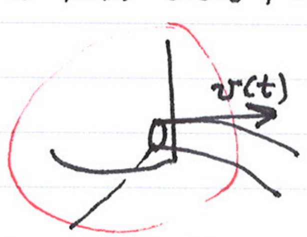
Want: characterize height $h(t)$ of fluid in tank. (above hole)



Cylindrical tank $\quad V(t) = A\,h(t)$

A ← cross sec. area

how fast does fluid leave.



$v(t)$ = veloc of fluid leaving hole
cm/s

cross sec. area of hole a (cm^2)

$a\,v(t) = cm^3/s$ = rate of flow out per unit time.

mass conservation:

rate of change of vol of fluid in container = rate volume leaves.

-ve since vol decr.

$$\frac{dV(t)}{dt} = -a\, v(t)$$

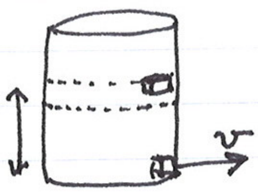
const.

$$\frac{d(A\, h(t))}{dt} = -a\, v(t)$$

$$* \quad \frac{dh}{dt} = -\frac{a}{A} v \quad \leftarrow \text{we need something about } v(t)$$

energy conservation:

poten energy = kinetic energy



$$\cancel{m} g\, h(t) = \frac{1}{2} \cancel{m} v^2(t)$$

$$\Rightarrow \quad \boxed{v(t)} = \sqrt{2g}\sqrt{h(t)}$$

sub this into *

$$\frac{dh}{dt} = \underbrace{-\frac{a}{A}\sqrt{2g}}_{\substack{\text{constant} \\ \shortparallel \\ k}}\sqrt{h}$$

O.D.E. $\quad \boxed{\frac{dh}{dt} = -k\sqrt{h}}$ $\quad$ Torricelli's law

I.C. $\quad h(0) = h_0$

separable, 1st order, nonlinear.

Soln: $\quad \frac{dh}{\sqrt{h}} = -k\, dt$

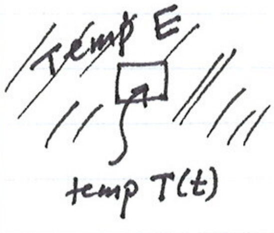
now integrate both sides...

⋮

HW.

Back to (applic of) 1st order $\underline{\text{linear}}$ eqns.

<u>Example 4</u> : Newton's Law of cooling:



"The rate of change of temp of object $T(t)$ is proportional to the difference of the ambient temp. E and" object temp."

ODE: $\boxed{\dfrac{dT}{dt} = k\,(E - T)}$ k ← constant $k > 0$ $\boxed{T(0) = T_0}$ I.C.

1st order, linear

$$\frac{dT}{dt} = \underbrace{kE}_{a} - \underbrace{k}_{b}T \quad \longleftrightarrow \quad \boxed{\frac{dy}{dt} = a - by}$$

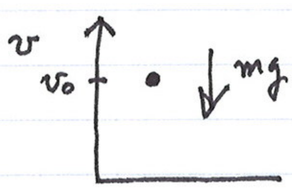
last time.

solve by separation

$$\frac{dT}{a - bT} = dt$$

or by integr. factor. ...

<u>Example 5</u>: Falling under force of gravity.



Newton's law:

Net force = rate of change of momentum.

"$F = ma$" (if mass constant)

$F = \dfrac{d}{dt}(mv)$ more general.

Simple case: mass = constant

but there may be friction, (drag force)

if no friction :
const mass

$$-\cancel{m}g = \cancel{m}\frac{dv}{dt} \qquad v(0) = v_0$$

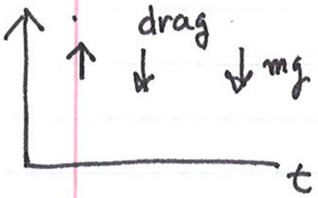
constant

$$dv = -g\,dt \qquad v(t) = v_0 - gt$$

with friction:

assumed frictional (drag force) proport. to vel.

$$m\frac{dv}{dt} = -mg - kv$$



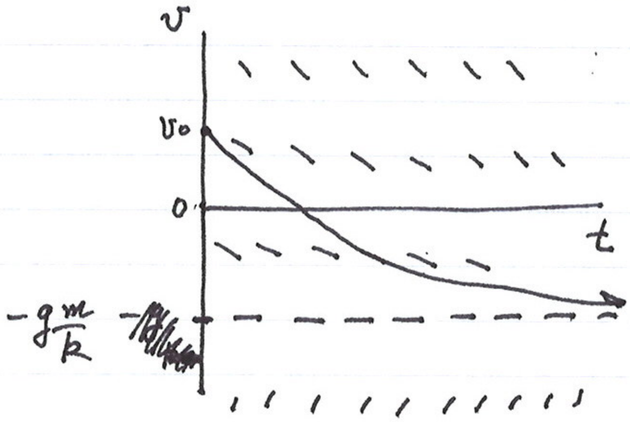
ODE: 1st order linear

$$\boxed{\frac{dv}{dt} = \underbrace{-g}_{a} - \underbrace{\frac{k}{m}}_{b} v}$$

$$\boxed{v(0) = v_0}$$

I.C.

$$\boxed{\frac{dy}{dt} = a - by}$$



there is a terminal veloc

$v(t) \to -\frac{gm}{k}$

as $t \to \infty$

Above, we used a direction field approach to see that velocity will approach $v_\infty \equiv -\frac{gm}{k}$

We could also solve the ODE by either separation of var. or by the integrating factor method