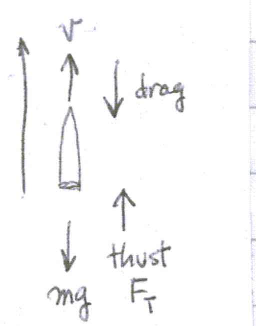
Details for Sept 15

<u>Rocket Equations</u> – with some simplifications

Newtons Law: Net force = Rate of change of momentum

$$F_{net} = \frac{d}{dt}(mv) \qquad (1)$$



m = mass of object (not constant! rocket uses up fuel)
V = velocity " " (upwards is taken as positive direction.)

$$(1) \Rightarrow \frac{d(mv)}{dt} = -mg - \text{drag force} + \text{thrust}$$

$$= \underbrace{-mg}_{\text{acts downwards}} \underbrace{- kv \quad + F_T}_{\text{acts upwards}}$$

here we made a simplifying assumption that drag is proportional to velocity (makes ODE linear)

- Assume mass of fuel is burned off at constant rate β, then mass of rocket satisfies (2) $\frac{dm}{dt} = -\beta$ $\qquad m(0) = m_0$ (initial mass with fuel)

$$(2) \Rightarrow m(t) = m_0 - \beta t \qquad \text{for } t \geq m_0/\beta$$

- Assume thust force is proportional to fuel burn rate β and speed of exhaust, here taken to be a constant (c)

$$F_T = c\beta \qquad (3)$$

Then (1, 3) $\Rightarrow$

$$\frac{d(mv)}{dt} = -mg - kv + c\beta$$

(product rule)

$$m\frac{dv}{dt} + v\underbrace{\frac{dm}{dt}}_{=-\beta} = -mg - kv + c\beta$$

$$m\frac{dv}{dt} - \beta v + kv = -mg + c\beta$$

$$\frac{dv}{dt} + \frac{(k-\beta)v}{m} = -g + \frac{c\beta}{m}$$

$$\frac{dv}{dt} + \frac{(k-\beta)}{(m_0 - \beta t)}v = -g + \frac{c\beta}{(m_0 - \beta t)}$$

Here we have put the ODE in standard form for integrating factor method.

And, Integrating factor is then

$$\mu(t) = \exp\left[\int \frac{k-\beta}{m_0-\beta t}\,dt\right] = \exp\left[(k-\beta)\,\frac{\ln(m_0-\beta t)}{-\beta}\right]$$

$$= \exp\left[\left(1-\frac{k}{\beta}\right)\ln(m_0-\beta t)\right]$$

$$= \exp\left[\ln\left((m_0-\beta t)^{\left(1-\frac{k}{\beta}\right)}\right)\right] = \boxed{(m_0-\beta t)^{1-\frac{k}{\beta}}}$$

$\mu(t)$

The ODE can be written

$$\frac{d}{dt}\big(v(t)\mu(t)\big) = \left(-g + \frac{C\beta}{m_0-\beta t}\right)\mu(t)$$

$$= -g\,(m_0-\beta t)^{1-\frac{k}{\beta}} + C\beta\,(m_0-\beta t)^{-\frac{k}{\beta}}$$

integrate both sides

$$v(t)\,\mu(t) = \int (\quad)\,dt + K$$

This is straightforward integration — Note: $\displaystyle\int (m_0-\beta t)^{Q}\,dt = -\frac{(m_0-\beta t)^{Q+1}}{\beta(Q+1)}$

$$v(t)\mu(t) = \frac{g\,(m_0-\beta t)^{2-\frac{k}{\beta}}}{\beta\left(2-\frac{k}{\beta}\right)} - \frac{C(m_0-\beta t)^{1-\frac{k}{\beta}}}{1-\frac{k}{\beta}} + K$$

$\mu(t)$

$(m_0-\beta t)^{1-\frac{k}{\beta}}$

Now divide both sides by $\mu(t)$

$$\Rightarrow v(t) = \frac{g(m_0-\beta t)}{\beta\left(2-\frac{k}{\beta}\right)} - \frac{C}{1-\frac{k}{\beta}} + K(m_0-\beta t)^{-\left(1-\frac{k}{\beta}\right)}$$

Those of us still willing to do some more work can use the initial condition to find the arbitrary constant K