Oct 6 Review

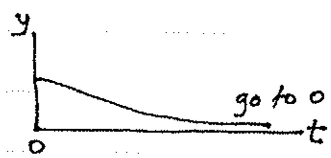
$\boxed{\text{First order Eqns.}}$

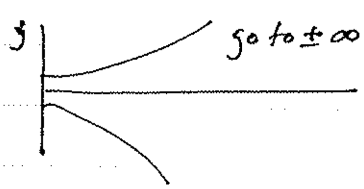
$$\frac{dy}{dt} = f(y,t)$$

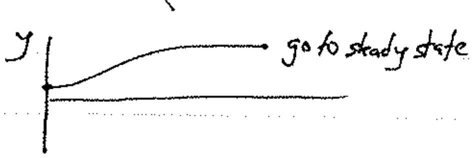
1. Linear
   - integrating factor

2. Nonlinear
   - separable?
   - direction field

Possible behaviour of solns in the case of $\frac{dy}{dt} = f(y)$







Note: in first order eqns we get no oscillations unless we have some input that "drives" the system periodically

$\boxed{\text{Second Order Eqns}}$

$$\frac{d^2y}{dt^2} = f\left(\frac{dy}{dt}, y, t\right)$$

Linear only

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \Leftarrow \text{homogeneous}$$

$ar^2 + br + c = 0$ char. eqn

roots $r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ (quadratic formula)

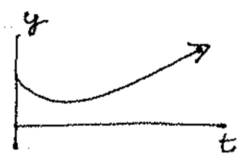
Cases:

- $b^2 - 4ac > 0$ $\quad r = r_1, r_2$ $\boxed{\text{real roots}}$

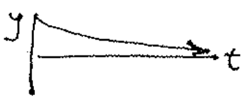
  $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

  if r_1 or $r_2 > 0$ soln grows

  $r_1, r_2 < 0$ soln decays

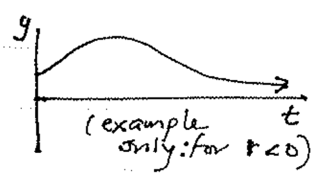


or



- $b^2 - 4ac = 0$ $\quad r = \frac{-b}{2a}$ $\boxed{\text{real repeated roots}}$

  $y(t) = c_1 e^{rt} + c_2 t e^{rt}$

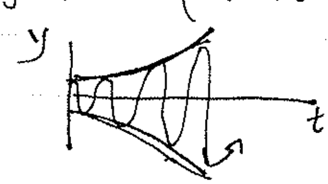


(example only: for $r<0$)

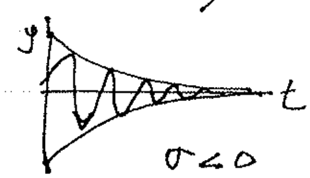
- $b^2 - 4ac < 0$ $\quad r = \sigma \pm i\omega$

  $\sigma = -\frac{b}{2a}$ $\quad \omega = \frac{\sqrt{|b^2-4ac|}}{2a}$

  $$y(t) = e^{\sigma t}\left(c_1\cos(\omega t) + c_2\sin(\omega t)\right)$$



$\sigma > 0$



$\sigma < 0$

More on first order

Direction fields:

Easy examples: (no time dependence, "autonomous")

$$\frac{dy}{dt} = y(1-y)(y+1)$$

- find where $\frac{dy}{dt} = 0$ to get locations of flat tangents

  e.g. $y = 0, 1, -1$

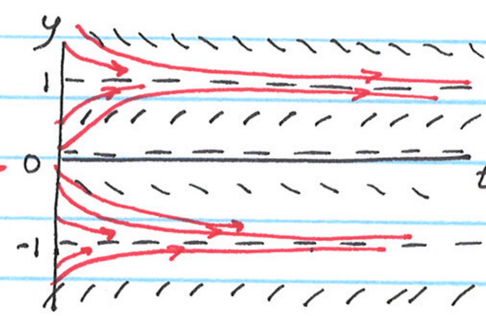
- plug in a few simple points
  to get sign of the expression for $\frac{dy}{dt}$
  (exact values less important)

e.g.

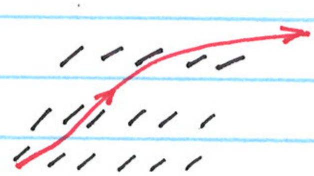
y	$\frac{dy}{dt}$	configuration
-2	$+$	/
-1	0	—
-0.5	$-$	\
0	0	—
0.5	$+$	/
1	0	—
2	$-$	\

- Assemble direction field

shown in red



- "solution curves" cannot
  cross these tangent vectors
  - cannot cross each other
  - only approach steady state values as $t \to \infty$



Second order linear NONHOMOGENEOUS Eqn

$$a y'' + b y' + c y = \underbrace{g(t)}_{\text{time dependent input}} \qquad \text{(N.H. ODE)}$$

• (1) Find soln of homog. problem

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

• (2) Compare $g(t)$ to $y_1(t)$, $y_2(t)$

- if distinct (not just constant multiple)
  guess particular soln related simply to $g(t)$

$$g(t) \text{ polynomial} \Longrightarrow Y_p(t) \text{ polynomial}$$

$$\left.\begin{matrix}\sin(\omega t)\\ \cos(\omega t)\end{matrix}\right\} \Longrightarrow A\sin(\omega t) + B\cos(\omega t)$$

$$e^{kt} \Longrightarrow A e^{kt}$$

- if related e.g. $g(t) = \text{constant} \cdot y_1$ or $\text{constant} \cdot y_2$
  multiply your guess by t (or t^2) to prevent this.

• (3) you have: $\rightarrow Y_p(t)$
now Compute $\begin{cases} Y_p'(t) \\ Y_p''(t) \end{cases}$ $\longrightarrow$ plug into (N.H. ODE)
determine A, B, etc. by equating coeffs of similar terms.

• (4) Write full soln

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y_p(t)$$

• (5) Use any initial conditions to find c_1, c_2. ← last step!! Note we first find A, B etc. in partic. soln.

General Background (but important!)

Suppose an equation as follows has to be satisfied for all t:

match

$$a t^2 + bt + c \sin(2t) + d \cos(3t) + h e^t = 2t - \cos 3t - e^t$$

match

This can only be true if "like" terms match exactly

$\Rightarrow$ $a = 0$ (equate coeffs of t^2 on both sides)

$b = 2$ " " t " "

$c = 0$ " " $\sin(2t)$ " "

$d = -1$ " " $\cos(3t)$ " "

$h = -1$ " " e^t " "

Calculus to remember:

product rule : $u = u(t)$, $v = v(t)$

- $$\frac{duv}{dt} = u\frac{dv}{dt} + v\frac{du}{dt}$$

integration by parts

$$\int u\,dv = uv - \int v\,du$$