Oct 4, 2010

Nonhomogeneous 2nd order ODE; Method of undetermined coeffs. con't

ODE: $ay'' + by' + cy = g(t)$ ← $g(t)$ is time-dependent input "forcing function"

solution

$y(t) = c_1 y_1(t) + c_2 y_2(t) + \underline{Y_p(t)}$

fundam. set of solns to homg. problem

particular soln to nonhomog. problem

Case 1:

If $g(t)$ is not some constant multiple of $y_1$ or $y_2$:

| | Form of forcing fn $g(t)$ | Your guess for $Y_p(t)$ | Comments |
|---|---|---|---|
| 1) | $t^2$ | $At^2 + Bt + C$ | Need lower order terms |
| 2) | $\sin(\omega t)$ or $\cos(\omega t)$ | $A\cos(\omega t) + B\sin(\omega t)$ | Need both sine + cosine |
| 3) | $e^{at}$ | $Ae^{at}$ | |
| 4) | $te^{at}$ | $(At + B)e^{at}$ | Need lower order terms |
| 5) | $e^{at}\sin(\omega t)$ or $e^{at}\cos(\omega t)$ | $e^{at}(A\cos(\omega t) + B\sin(\omega t))$ | ← both sine, cos |

Case 2: $g(t)$ proportional to $y_1(t)$ or $y_2(t)$ (e.g. cases 2-5)

Revise guess: multiply by factor of $t$ or $t^2$ so $Y_p(t)$ no longer proportional to either $y_1$ or $y_2$.

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Example: $y'' + 2y' + y = 3e^{-t}$ $\qquad g(t) = 3e^{-t}$

We saw (Sept 29) that soln to Hom. eqn is

$y(t) = c_1 e^{-t} + c_2 te^{-t}$ so $g(t)$ is "same as" $y_1(t) = e^{-t}$

Thus can't use partic. soln $Y_p(t) = Ae^{-t}$ nor $Ate^{-t}$

Need guess $Y_p(t) = At^2 e^{-t}$

General remarks :

- For other forms of forcing, e.g. $g(t) = \log(t+1)$
$g(t) = \csc(t)$

  can't use this method

- For sums of such forcing functions, e.g.

$$ay'' + by' + cy = g_1(t) + g_2(t) + \dots \quad \text{etc}$$

  can solve the individual problems — partic. solns

$$ay'' + by' + cy = g_1(t) \longrightarrow Y_{p_1}(t)$$
$$ay'' + by' + cy = g_2(t) \qquad Y_{p_2}(t)$$
$$\vdots$$

  and add up the results : $Y_p(t) = Y_{p_1}(t) + Y_{p_2}(t)$
  (see example : prob 23 p183 ...)

- If reach contradiction, your guess was inappropriate..
  try to revise guess

- the form of the homog. probl. solutions is important in
  forming reasonable guess.

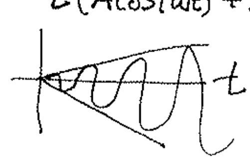
<u>Application:</u>

- the idea of <u>resonance</u> in forced vibrations is closely
  linked to the need to avoid $Y_p(t)$ duplicating the
  $\underbrace{\text{soln's of homog. problem}}$.

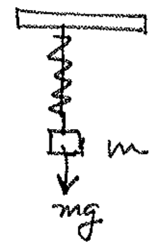
e.g: $\underbrace{c_1\cos(\omega t) + c_2\sin(\omega t)}$

Forced by
$g(t) = C\cos(\omega t)$
or
$C\sin(\omega t)$

$Y_p(t) =$
$t\,(A\cos(\omega t) + B\sin(\omega t))$



# Forced Undamped spring-mass system




Recall 2nd order, linear ODE for spring-mass system:

$$my'' + cy' + ky = 0$$

unloaded ("unforced") spring

$m$ = mass
$c$ = damping coeff
$k$ = spring constant
} constants, $\geq 0$
$y(t)$ = vertical displacement from ~~horiz~~ equil. position

Now consider case of negligible damping ($c \approx 0$), with applied force:

$$my'' + ky = g(t) \qquad \text{suppose } g(t) = P\cos(wt)$$

Find solution $y(t)$ given $y(0)=0$ $y'(0)=0$

Soln: • First: homog. problem: $my'' + ky = 0$

char. eqn: $mr^2 + k = 0$ $\qquad r^2 = -\frac{k}{m}$, $r = \pm i\sqrt{\frac{k}{m}}$

Soln: $y(t) = c_1 y_1(t) + c_2 y_2(t)$ where $y_1(t) = \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

$y_2(t) = \sin\left(\sqrt{\frac{k}{m}}\, t\right)$

• Now find particular solution to nonhom problem using the method of Undetermined coefficients.

nonhom prob: $my'' + ky = \underbrace{P\cos(wt)}$

$\Rightarrow$ guess soln $Y_p(t) = A\cos(wt) + B\sin(wt)$

to plug into eqn we first need all these derivs:

$Y_p'(t) = -Aw\sin(wt) + Bw\cos(wt)$

$Y_p''(t) = -Aw^2\cos(wt) - Bw^2\sin(wt)$

Now plug $Y_p(t)$ and its derivs' into nonhomog. ODE:

$$my'' \qquad + ky \qquad = P\cos(\omega t)$$

$$m\left(-A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)\right) + k\left(A\cos(\omega t) + B\sin(\omega t)\right) = P\cos(\omega t)$$

For this to be true for all $t$, the coeffs of (each of) sine and cosine should be equal on both sides. Sort the terms on each side:

$$\cos(\omega t)\left[-A\omega^2 m + kA\right] + \sin(\omega t)\underbrace{\left[-B\omega^2 m + kB\right]}_{\overset{\shortparallel}{0}\ \text{(no sines on other side)}} = \cos(\omega t)\left[P\right]$$

has to match

We find that

$$-A\omega^2 m + kA = P \qquad \text{to match coeffs of cosine}$$

$$-B\omega^2 m + kB = 0 \qquad \text{" " " " sine}$$

Since $m, k > 0$ we conclude $B = 0$

$$A(k - \omega^2 m) = P$$

$$\Rightarrow A = \frac{P}{(k - \omega^2 m)} \quad \Leftarrow \boxed{\text{provided } \frac{k}{m} \neq \omega^2}$$

particular soln:

$$Y_p(t) = \frac{P}{(k - \omega^2 m)}\cos(\omega t)$$

general soln:

$$y(t) = c_1\cos\left(\sqrt{\frac{k}{m}}\,t\right) + c_2\sin\left(\sqrt{\frac{k}{m}}\,t\right) + \overbrace{\frac{P}{k - \omega^2 m}\cos(\omega t)}$$

LAST Step:

- Use initial conditions to find the constants $c_1$ and $c_2$

$$y(0) = 0 \qquad y'(0) = 0$$

$$\boxed{\text{Find } y'(t) = -c_1\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right) + c_2\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}\,t\right) - \frac{P\omega}{k - \omega^2 m}\sin(\omega t)}$$

$$0 = y(0) = c_1 + \frac{P}{k - \omega^2 m} \qquad \Rightarrow \qquad c_1 = \frac{-P}{k - \omega^2 m}$$

$$0 = y'(0) = c_2\sqrt{\frac{k}{m}}\cdot 1 \qquad \Rightarrow \qquad c_2 = 0$$

- Final answer:

$$\boxed{y(t) = \frac{P}{k - \omega^2 m}\left(\cos(\omega t) - \cos\left(\sqrt{\frac{k}{m}}\,t\right)\right)}$$

Consider an undamped spring-mass system ($c=0$) with forcing

$$my'' + ky = P\cos(\omega t)$$

Examples: (first one was in-class work)

① $m=1$ $k=4$ $\omega=3$ $P=5$ $k-\omega^2 m = 4-9=-5$

frequency of unforced system $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{4} = 2$ ← not same as forcing frequency

soln to hom. problem $y(t) = C_1\cos 2t + C_2 \sin 2t$

particular soln: (after a lot of algebra) $Y_p = \frac{5}{4-9}\cos(3t) = -\cos(3t)$

full/gen'l soln:

$$y(t) = C_1 \cos 2t + C_2 \sin 2t - \cos(3t)$$

soln to IVP: (after using initial conds) $y(t) = -\left(\cos(3t) - \cos(2t)\right)$

Method is same as general case on previous pages.

Another one to try

② $m=1$ $k=9$ $\omega=2$ $P=10$ $k-\omega^2 m = 9-4=5$

$$\omega_0 = \sqrt{\frac{9}{1}} = \sqrt{9} = 3$$

hom soln: $y(t) = C_1\cos(3t) + C_2\sin(3t)$

partic. soln: $Y_p(t) = \frac{10}{5}\cos(2t) = 2\cos(2t)$

gen'l soln: $y(t) =$ sum of these

soln to IVP:

$$y(t) = 2\left(\cos(2t) - \cos(3t)\right)$$

Case of Resonance:

Suppose forcing frequency is <u>SAME</u> as natural frequency of the spring-mass system, i.e.

$$(*) \qquad my'' + ky = P\cos(\tilde{\omega} t) \qquad \tilde{\omega} = \sqrt{\frac{k}{m}} = \omega$$

Then previous guess for $Y_p(t)$ will not work since $k - \tilde{\omega}^2 m = 0$ (A is ~~a~~ division by zero).

We revise guess:

Find derivs of $Y_p(t)$ (messy)

$$Y_p(t) = t\left[A\cos(\omega t) + B\sin(\omega t)\right]$$

$$Y_p'(t) = \left[A\cos(\omega t) + B\sin(\omega t)\right] + t\left[-\omega A\sin(\omega t) + \omega B\cos(\omega t)\right]$$

$$= (A + \omega t B)\cos(\omega t) + (B - \omega t A)\sin(\omega t)$$

$$Y_p''(t) = -(A + \omega t B)\omega\sin(\omega t) + (B - \omega t A)\omega\cos(\omega t) + \omega B\cos(\omega t) - \omega A\sin(\omega t)$$

$$= -(2A + \omega t B)\omega\sin(\omega t) + (2B - \omega t A)\omega\cos(\omega t)$$

plug into ODE $(*)$

$$\tilde{\omega} = \omega = \sqrt{\frac{k}{m}}$$

$$m\left[-(2A + \omega t B)\omega\sin(\omega t) + (2B - \omega t A)\omega\cos(\omega t)\right] + k\left[tA\cos(\omega t) + tB\sin(\omega t)\right] = P\cos(\omega t)$$

Sort terms:

$$\sin(\omega t)\left[-2mA\omega\right] + t\sin(\omega t)\Big[\overbrace{-m\omega^2 B + kB}^{0 \text{ because } \omega^2 = \frac{k}{m}}\Big]$$

$$+ \cos(\omega t)\left[2Bm\omega\right] + t\cos(\omega t)\Big[\overbrace{-m\omega^2 A + kA}^{0}\Big] = P\cos(\omega t)$$

has to match

$$\Rightarrow A = 0 \qquad 2Bm\omega = P \qquad B = \frac{P}{2m\omega}$$

$$Y_p(t) = \frac{P}{2m\omega}\, t\sin(\omega t)$$

Gen'l soln

$$y(t) = c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{P}{2m\omega}\, t\sin(\omega t) \qquad \boxed{\omega = \sqrt{\frac{k}{m}}}$$

After more tedious algebra, we can find that $\boxed{y(t) = \left(\frac{P}{2m\omega}\right) t\sin(\omega t)}$ ← once we use Initial Conds

Example: $y'' + y = \cos(t)$

homog. pr: $r^2 + 1 = 0$ $\quad r = \pm i$ $\quad \Rightarrow \quad \omega = 1$ is natural freq.

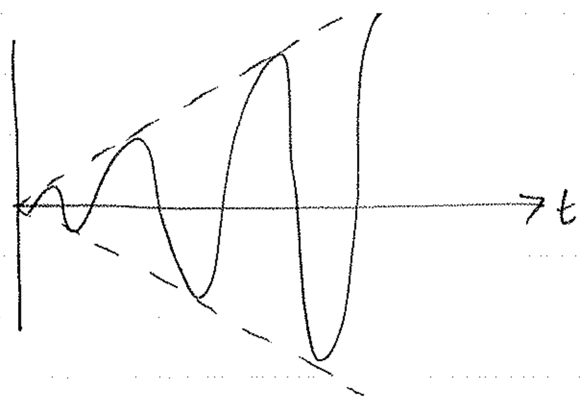
$y(t) = c_1 \cos t + c_2 \sin t$

$$Y_p(t) = t\big(A\cos(t) + B\sin(t)\big)$$

Find $A = 0$ $\quad B = \frac{1}{2}$ (Algebra omitted)

$$y(t) = c_1\cos(t) + c_2\sin(t) + \frac{1}{2}t\sin(t)$$

$$\left.\begin{array}{l} y(0)=0 \Rightarrow c_1 = 0 \\ y'(0)=0 \Rightarrow c_2 = 0 \end{array}\right\} y(t) = \frac{1}{2}t\sin(t)$$




oscillations keep growing.

Amplitude $\sim \frac{t}{2}$

$\Rightarrow$ resonance.