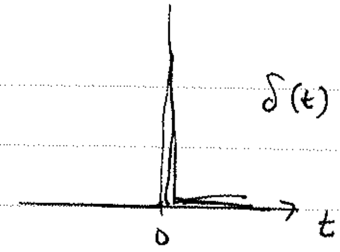
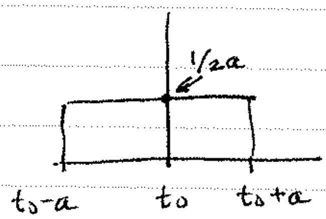
The Dirac delta function : "unit impulse"



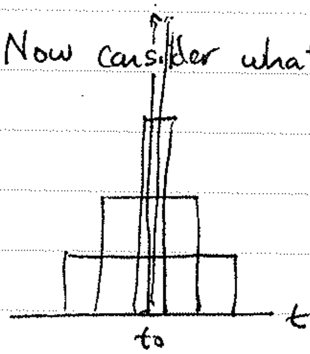
We will define this as a limiting form of a function as follows:



let $t_0 > 0$ Then we first define

$$\delta_a(t-t_0) = \begin{cases} \frac{1}{2a} & t_0 - a \le t \le t_0 + a \\ 0 & \text{otherwise} \end{cases}$$

Now consider what happens as we let the "width" $a \to 0$:



Define

$$\boxed{\delta(t-t_0) = \lim_{a \to 0} \delta_a(t-t_0)}$$

This is the 'unit impulse' or Dirac delta fn. (it is actually a 'generalized' function, called a distribution.)

- $\delta(t-t_0)$ is infinite at $t=t_0$
- $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$
- $\int_{-\infty}^{\infty} \delta(t-t_0) f(t)\,dt = f(t_0)$

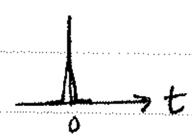
Q: What is the Laplace transform of $\delta(t-t_0)$?

A: $$\mathcal{L}\{\delta(t-t_0)\} = \int_0^{\infty} \delta(t-t_0)\, e^{-st}\,dt = e^{-st_0}$$

(can also be shown more rigorously using the definition above, limits, and using L'Hopital's rule).

Conclude: $$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}}$$

Corollary: $$\mathcal{L}\{\underbrace{\delta(t)}_{\delta \text{ fn at } t=0}\} = e^{-s\cdot 0} = 1$$



Solving ODE's with impulsive inputs.

Example (1) $y' - 3y = \delta(t-2) \quad y(0)=0$

$$\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-2)\}$$
$$(sF(s) - \underbrace{y(0)}_{=0}) - 3F(s) = e^{-2s}$$
$$(s-3)F(s) = e^{-2s}$$
$$F(s) = \frac{1}{(s-3)} e^{-2s}$$

Invert:

$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3} e^{-2s}\right\} = u_2(t) f(t-2)$ where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$

(leads to the shift and step $t \to t-2$)

so

$$y(t) = u_2(t) e^{3(t-2)} = H(t-2) e^{3(t-2)}$$

(The two alternate notations we are using)

Example (2) $y'' + y = \delta(t-2\pi) \quad y(0)=0 \quad y'(0)=1$

$$s^2F(s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=1} + F(s) = e^{-2\pi s}$$

$$F(s)(s^2+1) = 1 + e^{-2\pi s}$$
$$F(s) = \frac{1}{s^2+1} + \frac{e^{-2\pi s}}{s^2+1}$$

Invert: $y(t) = \mathcal{L}^{-1}\left\{\left(\frac{1}{s^2+1}\right)\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\}$

Same as $\sin(t)$

$$= \sin(t) + H(t-2\pi)\sin(t-2\pi) = \sin(t)\left[1 - H(t-2\pi)\right]$$

Transform of periodic function : suppose $f(t)$ is periodic, period T

so $f(t+T) = f(t)$

Then $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_{t=T}^{t=\infty} f(t)e^{-st}dt$ ← let $t = u+T$

$= \quad '' \quad + \int_{u=0}^{\infty} f(u+T)\,e^{-s(u+T)}du$

$= \quad '' \quad + e^{-sT}\int_0^\infty \underbrace{f(u+T)}_{=f(u) \text{ by periodicity}}e^{-su}du$

$$\mathcal{L}\{f(t)\} = \int_0^T f(t)e^{-st}dt + e^{-sT}\mathcal{L}\{f(t)\}$$

Solve for $\mathcal{L}\{f(t)\}$

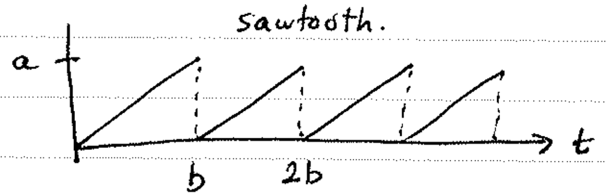
$$\mathcal{L}\{f(t)\}(1-e^{-sT}) = \int_0^T f(t)e^{-st}dt$$

Conclude: for $f(t)$ periodic with period T →

$$\boxed{\mathcal{L}\{f(t)\} = \frac{1}{(1-e^{-sT})}\int_0^T f(t)\,e^{-st}\,dt}$$

Example:

sawtooth.



Find $\mathcal{L}\{f(t)\}$ where

$$f(t) = \frac{a}{b}(t - nb) \qquad nb \le t \le b(n+1)$$

slope of each segment: $\frac{a}{b}$

x intercept: nb

period: $T = b$

Find $I = \int_0^T e^{-st}f(t)\,dt = \int_0^b e^{-st}\cdot\frac{a}{b}(t)\,dt$

$= \frac{a}{b}\int_0^b te^{-st}dt$ ← intgr. by parts $u = t \quad du = dt$, $dv = e^{-st}dt \quad v = \frac{e^{-st}}{-s}$

$= \frac{a}{b}\left[-\frac{b}{s}e^{-sb} + \frac{1-e^{-sb}}{s^2}\right]$

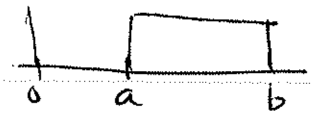
use the result for periodic fns.

$$\mathcal{L}\{f(t)\} = \frac{1}{(1-e^{-sb})}\int_0^b f(t)e^{-st}dt$$

$$= \frac{1}{(1-e^{-sb})}\cdot -\frac{a}{s}e^{-sb} + \frac{a}{bs^2}$$

$$= \frac{a}{s}\left(\frac{1}{bs} - \frac{e^{-sb}}{1-e^{-sb}}\right)$$

Find the charge $q(t)$ in an LRC circuit with switch



$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E_0\left[H(t-a) - H(t-b)\right] \quad \text{switch}$$

$q(0)=0$
$q'(0)=0$

As we may be interested in various inputs, let us first set up a more general example, with $V(t)$ as applied voltage.

Laplace transform, applied to both sides:

$$L\left(s^2F(s) - s\underbrace{q(0)}_{0} - \underbrace{q'(0)}_{0}\right) + R\left(sF(s) - \underbrace{q(0)}_{0}\right) + \frac{1}{C}F(s) = \mathcal{L}\{V(t)\}$$

$$F(s)\left(s^2L + sR + \frac{1}{C}\right) = \mathcal{L}\{V(t)\}$$

$$F(s) = \frac{1}{\left(s^2L + sR + \frac{1}{C}\right)}\mathcal{L}\{V(t)\}$$

Example (I) Suppose $R \approx 0$, $L = 0.25$, $C = 4$ $\quad V(t) = 10\left(H(t-2) - H(t-5)\right)$

let $q(0)=0$; $q'(0)=0$ as above $\quad$ Then $1/C = 0.25$

$$F(s) = \frac{1}{0.25(s^2+1)}\mathcal{L}(V(t)) = 4\,\frac{1}{(s^2+1)}\cdot 10\left(\frac{e^{-2s} - e^{-5s}}{s}\right)$$

$$= 40\,\frac{1}{s^2+1}\cdot\frac{1}{s}\left(e^{-2s} - e^{-5s}\right)$$

Partial Fractions:

$$\frac{1}{s^2+1}\cdot\frac{1}{s} = \frac{As+B}{s^2+1} + \frac{\tilde{C}}{s} = \frac{As^2 + Bs + \tilde{C}s^2 + \tilde{C}}{(s^2+1)s}$$

$$= -\frac{1\cdot s}{s^2+1} + \frac{1}{s}$$

$A + C = 0$ ← s^2 term
$B = 0$ $\quad$ s term
$\tilde{C} = 1$ ← const. term
$\Rightarrow A = -1,\ B = 0,\ C = 1$

Example (I) cont'd

$$F(s) = 40\left[\left(\frac{-s}{s^2+1} + \frac{1}{s}\right)e^{-2s} - \left(\frac{-s}{s^2+1} + \frac{1}{s}\right)e^{-5s}\right]$$

$$\mathcal{L}^{-1}\{F(s)\} = u_2(t)\left(-\cos t + 1\right)\Big|_{t\to t-2} - u_5(t)\left(-\cos t + 1\right)\Big|_{t\to t-5}$$

$$y(t) = u_2(t)\left[1-\cos(t-2)\right] - u_5(t)\left[1-\cos(t-5)\right]$$

$$= u_2(t) - u_5(t) - u_2(t)\cos(t-2) + u_5(t)\cos(t-5)$$

Example (II) Solve the following IVP for a spring-mass system with mass $m=2$, spring constant $k=8$, driv. freq. $\omega=2$

(starting from rest at equilibrium) $my'' + ky = F(t) = 2\cos(\omega t)$ $\quad y(0)=y'(0)=0$

$$2y'' + 8y = 2\cos(2t)$$

$$y'' + 4y = \cos(2t)$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\cos(2t)\}$$

$$\left[s^2F(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\right] + 4F(s) = \frac{s}{s^2+4}$$

$$(s^2+4)F(s) = \frac{s}{s^2+4}$$

$$F(s) = \frac{1}{s^2+4}\cdot\frac{s}{s^2+4} = \frac{s}{(s^2+4)^2}$$

Inverting is left as an exercise.

Remark: use $\mathcal{L}\{t\sin t\} = -\frac{d}{ds}F(s)$ where $F(s) = \mathcal{L}\{\sin t\}$