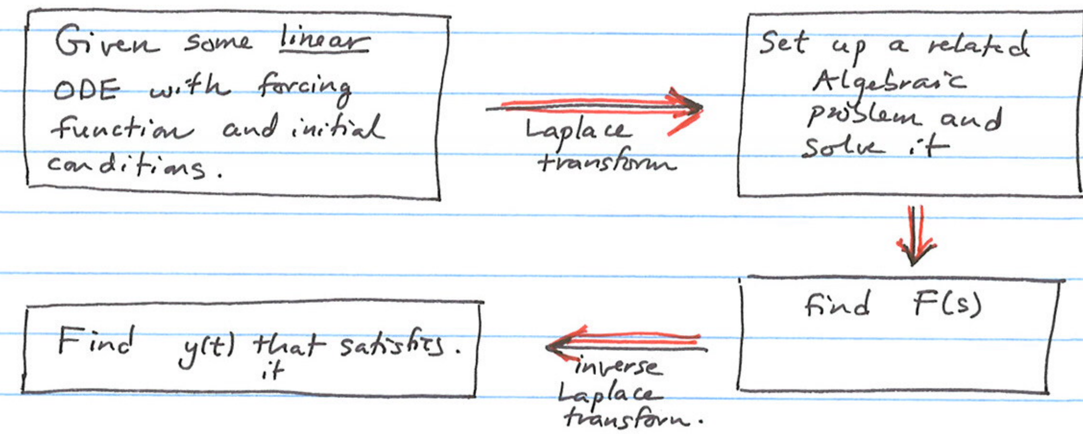
Introduction to the Laplace Transform

Problem:



Advantages:

- Easy to treat all kinds of inputs (impulses, step functions, as well as sines, cosines, exponentials etc.)
- obtain all needed constants at once (e.g. uses initial cond's, and finds particular + homog. soln all at once).

New concepts:
- What is a transform (also denoted "linear operator")
- Uses and defn of improper integrals
- "acceptable" and other functions

New skills:
- Integration by parts
- limits such as e^{-at}, $t^n e^{-at}$ for $t \to \infty$
- recursions
- partial fractions.

Defn: Given a function $y = f(t)$, we define its Laplace transform as

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt} \quad \leftarrow \text{integrate over } t$$

Note: (1) This is an "improper" integral (i.e. over $0 \le t \le \infty$)
due caution required!

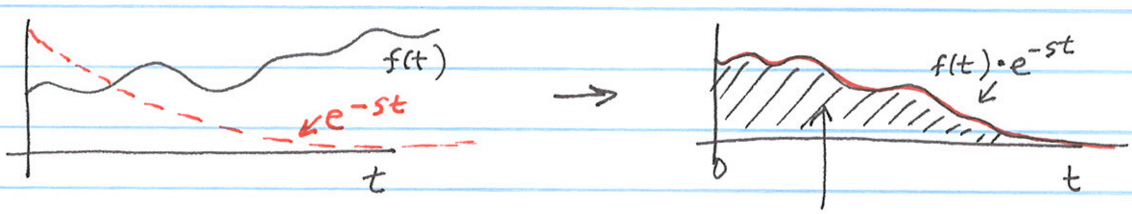
(2) Result does not depend on t! but it depends on s.
Call it $F(s)$
(i.e. Let $F(s) \equiv \mathcal{L}\{f(t)\}$

(3) $\mathcal{L}$ is a linear operator (HW 5)
that is $\mathcal{L}\{a f(t) + b g(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$

We will shortly see how this Laplace transform is useful.

In order that the Laplace transform of a function exists (i.e. has a meaning, or, restated, that the improper integral converges) it has to be true that

- $f(t)$ cannot "blow up" anywhere
- $f(t)$ cannot grow too quickly as $t \to \infty$



Area under this curve is
$$\int_0^\infty e^{-st} f(t)\, dt \equiv \mathcal{L}\{f(t)\} = F(s)$$
it has to be finite if $F(s)$ is to exist.

Example of what the Laplace Transform is good for:

→ Solve the initial value problem

$$\boxed{y'' + 3y' + 2y = 0} \qquad y(0) = 1 \qquad y'(0) = 0$$

To do this use: (facts we'll show later) HW5

(To be proven in later class or HW)

→ if $F(s) = \mathcal{L}\{y(t)\}$

then:

$$\mathcal{L}\{y'(t)\} = sF(s) - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2F(s) - sy(0) - y'(0)$$

convert ODE to transformed eqn

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{0\} = 0$$

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 0$$

$$\left[s^2F(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0}\right] + 3\left[sF(s) - \underbrace{y(0)}_{1}\right] + 2F(s) = 0$$

use initial conditions in above step

$$s^2F(s) + 3sF(s) + 2F(s) - s - 3 = 0$$

$$(s^2 + 3s + 2)F(s) = s + 3$$ ← We have reduced it to an Algebraic problem: Find $F(s)$

$$F(s) = \frac{s+3}{(s^2+3s+2)}$$ ← need to simplify to use look-up table

algebra (Partial Fractions)

$$= \frac{A}{(s+1)} + \frac{B}{(s+2)}$$ ← Find A, B partial fractions

details: (next page) →

$$= \frac{2}{(s+1)} - \frac{1}{(s+2)}$$

use Table to find inverse Laplace transform

$$\boxed{y(t) = 2e^{-t} - e^{-2t}}$$

see p 317 Boyce + Diprima 9th ed.

Details of Partial Fractions:

Partial Frac: $F(s) = \dfrac{s+3}{s^2+3s+2} = \dfrac{s+3}{(s+2)(s+1)} = \dfrac{A}{s+2} + \dfrac{B}{s+1}$

$= \dfrac{A(s+1) + B(s+2)}{(s+2)(s+1)} = \dfrac{s(A+B) + (A+2B)}{(s+2)(s+1)}$

match "like" terms

s^1 term: $A+B=1$

s^0 term: $A+2B=3$

$B=2, \quad A=-1$

$\Rightarrow F(s) = \dfrac{-1}{(s+2)} + \dfrac{2}{(s+1)}$

Before going on, quick review of improper integrals

Defn Let $I = \int_a^\infty g(t)\,dt$ then I is an improper integral

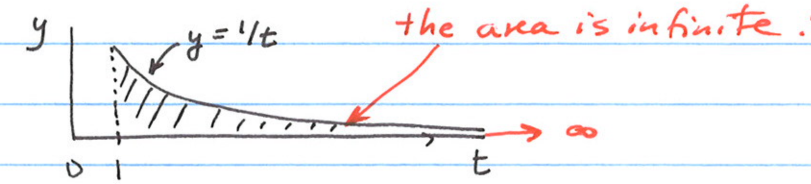
and we understand it as $\lim_{x \to \infty} \int_a^x g(t)\,dt$

We say that "I exists" or "the improper integral converges" if this limit exists. Otherwise we say "the improper integral diverges".

Examples:

① $$\int_1^\infty \frac{1}{t}\,dt = \ln t \Big|_1^\infty = \lim_{x\to\infty} \ln x - \underbrace{\ln 1}_{=0} = \lim_{x\to\infty} \ln x = \infty$$

THIS INTEGRAL DIVERGES.!!



② $$\int_1^\infty \frac{1}{t^p}\,dt \quad \begin{cases} \text{converges for } p > 1 \\ \text{diverges for } p \le 1 \end{cases}$$ (HW5)

③ $$\int_0^\infty e^{-st}\,dt = \frac{e^{-st}}{-s}\Big|_0^\infty = -\frac{1}{s}\left[\lim_{t\to\infty} e^{-st} - e^0\right]$$

$$= -\frac{1}{s}\left[0 - 1\right] = \frac{1}{s} \quad \text{for } s > 0$$

where we have use the following important fact: For $s > 0$ $\lim_{t\to\infty} e^{-st} = 0$

Remark: later, we'll use related limits:

For $s > 0$ $\lim_{t\to\infty} t^n e^{-st} = 0$