Nov 22, 2010

Repeated Roots in ODE systems

Math 265

We want to solve $\frac{d\vec{x}}{dt} = M\vec{x}$, a system of ODEs.

Consider the case that a matrix M has repeated (identical) eigenvalues. (e.g. for M a 2×2 matrix, this happens when $\beta = \mathrm{Tr}\, M$, $\gamma = \det M$, and $\beta^2 = 4\gamma$)

Then there are two cases to consider.

(1) Sometimes we can find linearly independent eigenvectors $\vec{v}_1, \vec{v}_2$ corresponding to these eigenvalue. If so, then gen'l soln is

$$\vec{x}(t) = c_1 \vec{v}_1 e^{rt} + c_2 \vec{v}_2 e^{rt}$$

(This is the easy case). Note: this is always true if M is Hermitian.

See Example 1

(2) In some cases, there is only one eigenvector $\vec{v}_1$ corresponding to r. Then we need to work a little harder to build up the general soln. (i.e. to find a second linearly indep. soln.) we have only $\vec{x}_1(t) = \vec{v}_1 e^{rt}$ and need to find $\vec{x}_2(t)$ so that $\vec{x}_1, \vec{x}_2$ form a funamental set.

See Example 2.

Example 1

$$\frac{d\vec{x}}{dt} = M\vec{x} \qquad M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Hermitian matrix, diagonal $\Rightarrow$ eigenvalues on diagonal

$$\left(\underset{(\mathrm{Tr}\,M)}{\beta = 4} \qquad \gamma = \det M = 4 \right)$$

eigenvalues $r_1 = 2, 2$ repeated

eigenvectors $\begin{pmatrix} 2-2 & 0 \\ 0 & 2-2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

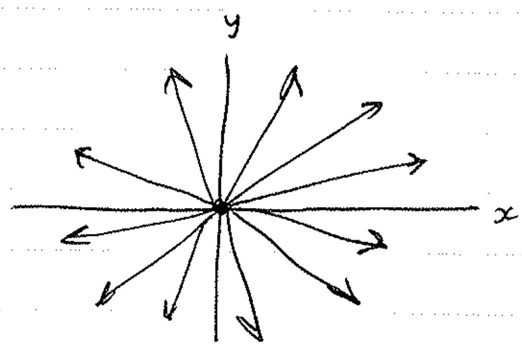
v_1 and v_2 can be anything, e.g. $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

General soln $\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t} = e^{2t}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

These are lin. indep

Check:

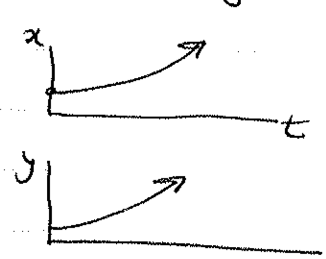
Wronskian: $W = \det \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} = e^{4t} \neq 0$



This is called an UNSTABLE NODE

Behaviour of the solns in the xy plane is shown here. For any initial point (x_0, y_0), find that solns move along straight lines. Since eigenval. is real, positive, the trajectories move towards increasing values of x, y.

Equivalent picture:



exponential growth.

Remark, in Example 1, the matrix is Hermitian, defined below:

Hermitian (or self-adjoint) matrix: p 382 B+D.

$$A^* = A \quad \text{i.e. } \bar{a}_{ji} = a_{ij}$$

Subclass: Real, symmetric matrices s.t. $A^T = A$

properties:

- Eigenvals real
- there are n linearly indep eigenvec.
- eigenvec's for distinct (non repeated) eigenvalues are $\perp$

Example $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ p 381

$r_1 = 2,\ r_2 = -1,\ r_3 = -1$

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ Note distinct, linearly indep eigenvectors

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(Hermitian and real symmetric matrices have particularly nice properties, as shown in the example, though in 2×2 matrices it is a bit trivial.)

Sys. of Eqns with Repeated Roots in Case 2 (the more complicated case)

Example 2: $\frac{d\vec{x}}{dt} = M\vec{x}$ $\quad M = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$ $\quad \beta = \text{Tr}\,M = -6$

$\gamma = \det M = -27+36 = 9$

$$r^2 + 6r + 9 = 0$$

$$(r+3)^2 = 0 \qquad r = -3, -3$$

repeated root ($\equiv$ eigenvalue of multiplicity 2)

Let us ask how many eigenvectors there are corresponding to this eigenvalue

$$(M - rI)\cdot\vec{v} = \begin{pmatrix} 3+3 & -18 \\ 2 & -9+3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 6 & -18 \\ 2 & -6 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} 6v_1 - 18v_2 &= 0 \\ 2v_1 - 6v_2 &= 0 \end{aligned}\right\} \rightarrow \quad v_1 = 3v_2$$

get only one eigenvector, $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and, so, only one soln to ODE system so far:

soln: $\vec{x}_1(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}$

We want to find a second soln (lin. indep. from the above) so we can form the general soln.

Attempt 1: Try $\vec{x}_2(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-3t}$ will it work?

$$\vec{x}_2{}'(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}\left(e^{-3t} + t(-3)e^{-3t}\right) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}(1-3t)e^{-3t}$$

plug into ODE: $\vec{x}_2{}' = M\vec{x}_2 \Rightarrow$

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix}(1-3t)\cancel{e^{-3t}} = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}\cdot\begin{pmatrix} 3 \\ 1 \end{pmatrix} t\,\cancel{e^{-3t}}$$

$\leftarrow$ This has to be true for all t.

This is clearly not possible, since, for example, for $t=0$ we find that $\begin{pmatrix} 3 \\ 1 \end{pmatrix}\cdot(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ CONTRADICTION

Even though the above guess is "reasonable", it DOES NOT WORK. NEED A DIFFERENT guess.

Attempt 2: $\vec{X}_2(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} e^{-3t}$ ← some unknown vector, whose components we will find.

$$\vec{X}_2'(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \left(e^{-3t} - 3te^{-3t} \right) - 3 \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} e^{-3t}$$

$$= e^{-3t} \begin{pmatrix} 3 - 9t - 3q_1 \\ 1 - 3t - 3q_2 \end{pmatrix}$$

plug into:

$$\vec{X}_2'(t) = M \vec{X}_2(t)$$

$$\cancel{e^{-3t}} \begin{pmatrix} 3-9t-3q_1 \\ 1-3t-3q_2 \end{pmatrix} = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix} \cdot \begin{pmatrix} 3t + q_1 \\ t + q_2 \end{pmatrix} \cancel{e^{-3t}}$$

$$= \begin{pmatrix} 3(3t+q_1) - 18(t+q_2) \\ 2(3t+q_1) - 9(t+q_2) \end{pmatrix} = \begin{pmatrix} -9t + 3q_1 - 18q_2 \\ -3t + 2q_1 - 9q_2 \end{pmatrix}$$

the terms multiplying t's cancel and we get

$$\begin{aligned} 3 - 3q_1 &= 3q_1 - 18q_2 \\ 1 - 3q_2 &= 2q_1 - 9q_2 \end{aligned} \quad \text{or} \quad \begin{aligned} 3 &= 6q_1 - 18q_2 \\ 1 &= 2q_1 - 6q_2 \end{aligned} \quad (**)$$

Eqs are redundant, so take any one, e.g. $1 = 2q_1 - 6q_2$

pick $q_1 = 1$ then $q_2 = 1/6$

So $\vec{X}_2(t) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 1 \\ 1/6 \end{pmatrix} e^{-3t}$ ← this is the second soln we needed

gen'l soln: $\vec{X}(t) = C_1 \vec{X}_1(t) + C_2 \vec{X}_2(t)$

$$= C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + C_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 1 \\ 1/6 \end{pmatrix} e^{-3t} \right]$$

Remark: It is a bit messier, but we can use the Wronskian to find that $\vec{X}_1, \vec{X}_2$ form a fundamental set of solns.

More <u>generally</u>: r repeated root, with only <u>one</u> eigenvector assoc., e.g.

$$\vec{X}_1 = \vec{v}_1 e^{rt} \quad \text{is one soln}$$

Then form a second soln as follows:

$$\text{Let } \vec{x}_2 = \vec{v}_1 t e^{rt} + \vec{v}_2 e^{rt} = e^{rt}(\vec{v}_1 t + \vec{v}_2)$$

$$\vec{X}_2' = e^{rt}(\vec{v}_1(1+rt) + r\vec{v}_2)$$

$$\cancel{e^{rt}}(\vec{v}_1(1+rt) + r\vec{v}_2) = M\cdot(\vec{v}_1 t + \vec{v}_2)\cancel{e^{rt}}$$

has to hold for all values of t so

(for $t=0$) $\quad \vec{v}_1 + r\vec{v}_2 = M\vec{v}_2$

$$\vec{v}_1 = M\vec{v}_2 - r\vec{v}_2$$

$$\boxed{\vec{v}_1 = (M - rI)\cdot\vec{v}_2}$$

The terms multiplying t:

$$\vec{v}_1 r = M\cdot\vec{v}_1$$

$$0 = (M - Ir)\cdot\vec{v}_1$$

← this just says that $r, \vec{v}_1$ are an eigenvalue — eigenvector pair. (we already know that)

Here is the new part. We must solve this for $\vec{v}_2$ in each case.

<u>Returning to Example 2</u>: — $\vec{v}_1 = \begin{pmatrix}3\\1\end{pmatrix}$, $r = -3$ so eqn in box is

$$\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}3+3 & -18\\ 2 & -9+3\end{pmatrix}\begin{pmatrix}q_1\\q_2\end{pmatrix}$$

$$\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}6q_1 - 18q_2\\ 2q_1 - 6q_2\end{pmatrix}$$

which is precisely the system (**) we had arrived at.

How do we sketch this type of soln? In Example 2, we found:

$$\vec{x}(t) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 1 \\ 1/6 \end{pmatrix} e^{-3t} \right]$$

- If we have an initial value $\vec{x}(0) = \alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ i.e. any multiple of the eigenvector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, then (plug in $t=0$) we find $c_1 = \alpha$, $c_2 = 0$

  $\Rightarrow$ soln $\vec{x}(t) = \alpha \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-3t}$

  this a trajectory moving towards $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ along the direction $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

- All trajectories go towards the origin (since $e^{-3t} \to 0$ as $t \to \infty$)

- As $t \to \pm\infty$, the direction of the trajectories becomes more + more parallel to $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

  [This can be seen by finding $\vec{x}'(t)$ and noting that one component of that soln is proportional to t]